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## Correspondence

### Piecewise Approximation of Pictures Using Maximal Neighborhoods

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**Abstract**—Suppose that we are given a picture having approximately piecewise constant gray level. Each point  $P$  has a largest neighborhood  $N(P)$  that is entirely contained in one of the constant regions, and the set of maximal  $N(P)$ 's (i.e.,  $N(P)$ 's not contained in other  $N(P)$ 's) constitutes an economical description of the picture, generalizing the Blum "skeleton" or medial axis transformation. This description can be used to construct approximations to the picture (e.g., by discarding small  $N(P)$ 's). The picture can be smoothed, without excessive blurring, by averaging over each  $N(P)$ . By taking differences between pairs of touching maximal  $N(P)$ 's, the edges between the regions can be detected; since this edge detection scheme is not based on symmetrical detection operators, it is not handicapped when two adjacent regions differ greatly in size.

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**Index Terms**—Edge detection, image processing, medial axis transformation, picture processing, piecewise approximation, smoothing.

#### I. INTRODUCTION

This paper develops a general method of constructing piecewise approximations to a picture. The picture is assumed to be composed of a set of regions, each having approximately constant gray level (possibly noisy). Many types of real-world pictures seem to fit these assumptions reasonably well; two examples of such pictures are shown in Fig. 1.

The approximations are defined by sets of neighborhoods, each of which is contained in one of the regions, but is not contained in any other such neighborhood. (A more precise definition is given in Section II, and the implementation of the method is described in Section III.) For brevity, we shall refer to this type of approximation from now on as a SPAN (Spatial Piecewise Approximation by Neighborhoods).

The SPAN is a generalization of Blum's Medial Axis Transformation (MAT) [1]. The MAT was originally defined for two-valued pictures, and the definition was later extended [2], [3] to gray scale objects that have been explicitly segmented from their

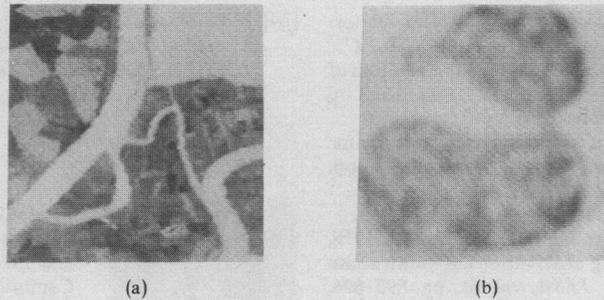


Fig. 1. Pictures used in SPAN experiments ( $127 \times 127$ , 64 gray levels). (a) Portion of a LANDSAT image. (b) White blood cell.

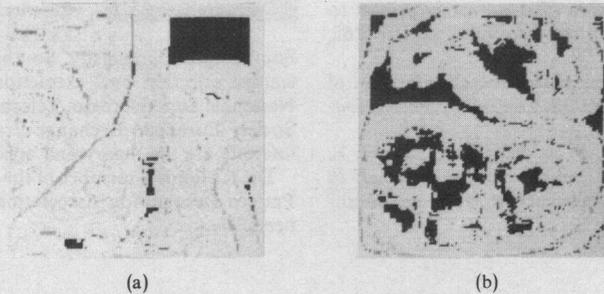


Fig. 2. SPAN neighborhood centers for Fig. 1(a), (b) displayed with gray levels proportional to their radii (radius 7 = black).

background. The SPAN provides another generalization that applies to unsegmented gray scale pictures. Like the MAT, it provides a compact (although approximate) representation of the picture; a method of constructing successively coarser approximations (e.g., by discarding small neighborhoods); and a basis for describing region shapes: branches on the SPAN correspond to lobes on the region.

Other approaches to piecewise approximation of pictures have been investigated by Pavlidis and his students [4]–[7]. Typically, such approaches begin with an initial partition of the picture into cells, and modify the partition by merging and splitting cells, and adjusting cell boundaries, while ensuring that a given approximation criterion remains satisfied on each piece of the partition. The SPAN approach provides an interesting alternative because, like the MAT, it provides a natural basis for defining successive approximations and structural region descriptions, as already pointed out. Moreover, the SPAN is constructed by a “parallel” (i.e., order-independent) process of neighborhood growing and nonmaximum suppression, which could be implemented very efficiently on a parallel array-processing computer.

## II. THE MAT AND THE SPAN

The MAT of a set  $S$  can be defined as the set of centers (and radii) of the maximal disks that are contained in  $S$  [1]. The “disks” need not be circular; when working with digital pictures, it is more convenient to use squares. It can be shown that the MAT is the set of points of  $S$  whose distances from  $\bar{S}$  are local maxima. The gray-weighted MAT [2], [3] allows  $S$  to have gray levels, and uses a “gray-weighted” distance to  $\bar{S}$ ; the details will not be given here.

The SPAN is defined analogously to the MAT, but does not require the given picture to be segmented into a set  $S$  and its complement; it assumes only that the picture consists of regions  $R$ , each of which has approximately constant gray level. As the examples will show, this restriction is not very strong in practice;

gradual gray level variations, noise, and texture (if it does not have high contrast) can all be tolerated.

Let  $(x, y)$  be a point of one of the constant regions  $R$ , and let  $N_r(x, y)$  be the disk of radius  $r$  centered at  $(x, y)$ . We would like to find the largest  $r = r(x, y)$  such that  $N_r(x, y)$  is entirely contained in  $R$ . Our approach will be to apply some simple statistical tests to the gray level population in  $N_r(x, y)$  in order to decide whether  $N_r(x, y)$  is contained in a single constant region or overlaps several of the regions. A related approach was used in [8] in an attempt to determine an optimal degree of smoothing to use at each point of a picture. In [9], an analogous method was used to find natural piecewise constant approximations of one-dimensional strings of data.

A number of statistical tests that can be used to determine  $r(x, y)$  were discussed in [10], [11], and results obtained using these tests were compared. In this paper we use only a simple multimodality test: we assume that the gray level in each region  $R$  is unimodally distributed, so that if  $N_r(x, y)$  has a multimodal distribution of gray levels, it cannot be contained in  $R$ .

To test for multimodality, we first smooth the neighborhood's gray level histogram by averaging over a 5-gray-level neighborhood of each histogram point. In the resulting smoothed histogram, changes in the sign of the slope that persist for three or more successive points are counted. Each such change from positive to negative represents a peak. A histogram with two or more peaks is taken to be multimodal.

In the applications described in the next section, a fixed set of  $r$ 's is used to reduce computational cost:  $r = 0, 1, \dots, 7$ . At each point  $(x, y)$ , we apply the multimodality test successively to these neighborhoods, starting with the largest size. The first size found not to be multimodal is the desired  $r(x, y)$ . If the sizes down to  $r = 2$  (corresponding to a 5-by-5 neighborhood) are all found to be multimodal, we use a special procedure to test size  $r = 1$ , since multimodality cannot be meaningfully measured on a 9-point distribution. Specifically, we apply a 3-by-3 gradient operator at

TABLE I  
NUMBERS OF MAXIMAL NEIGHBORHOODS AND SPAN NEIGHBORHOODS  
HAVING EACH RADIUS FOR FIG. 1(a), (b)

Figure	Radius	Number of Neighborhoods	Number of SPAN Neighborhoods
1a	7	981	981
	6	330	86
	5	624	183
	4	959	213
	3	1433	190
	2	2122	339
	1	3679	663
	0	<u>6001</u>	<u>863</u>
	Total	16129	3518
1b	7	2810	2810
	6	1216	205
	5	1395	126
	4	1597	90
	3	1742	94
	2	2048	126
	1	1619	34
	0	<u>3702</u>	<u>148</u>
	Total	16129	3633

$(x, y)$ ; if its value is low, we take  $r(x, y)$  to be 1, but if the value is high, we set  $r(x, y) = 0$  (the neighborhood is the point itself).

Once  $r(x, y)$  has been found in this way, let  $N(x, y)$  be the neighborhood of  $(x, y)$  that has radius  $r(x, y)$ . We discard  $N(x, y)$  if it is contained in  $N(u, v)$  for some other point  $(u, v)$ . The remaining set of maximal  $N(x, y)$ 's (or equivalently, their centers and radii) defines the SPAN of the given picture.

### III. EXAMPLES

Fig. 2 shows SPAN's constructed for the pictures in Fig. 1 using the method just described. The centers of the SPAN neighborhoods are displayed with gray levels proportional to their radii (black = radius 7). In some regions there are solid blocks of black points because these regions were larger than  $15 \times 15$ . Table I shows the number of maximal neighborhood having each radius.

One of the useful properties of Blum's MAT is that it can be used to reconstruct the original image subset, since this is just the union of the maximal neighborhoods in the MAT. More important, if neighborhoods of small sizes are omitted from the MAT, the reconstruction yields a simplified version of the original subset, with small parts deleted, but major parts (approximately) preserved. We now consider the corresponding problem of reconstructing an approximation to the original image from the SPAN.

The natural building blocks for such a reconstruction are the maximal neighborhoods  $N(x, y)$ , each filled in with a constant gray level equal to its average gray level on the original image. However, it is not immediately clear how these  $N(x, y)$ 's should be combined when they overlap (and have different gray levels). We have used the following rules of combination.

1) When maximal  $N(x, y)$ 's having different radii overlap, we use the gray level belonging to the one with the smaller radius. (Rationale: The ones with small radii are needed to provide fine image detail; they could not do this if they were overridden by the larger ones. Indeed, the large ones may pass the acceptance tests even when they slightly overlap neighboring regions, and this could cause the mean gray level of one region to be given to points of an adjacent region.)

2) When maximal  $N(x, y)$ 's having the same radius overlap, we (arbitrarily) use the maximum of their gray levels. (Alternatives would be to use the minimum or the average; results using these were compared in [11], but they were almost indistinguishable from the results using the maximum.) Thus, each point gets the maximum average gray level of the smallest maximal  $N(x, y)$ 's that contain it.

Fig. 3 shows steps in the reconstruction of the pictures of Fig. 1 using these rules of combination. We display the cumulative effects of combining the  $N(x, y)$ 's having different radii, beginning with the largest radius. We see that reasonable approximations are obtained when small radii are included.

### IV. APPLICATIONS: SMOOTHING AND EDGE DETECTION

Since each neighborhood  $N(x, y)$  is contained within one of the regions  $R$ , we should be able to smooth the picture without blurring the edges of the regions by replacing the gray level at  $(x, y)$  by the average gray level measured over  $N(x, y)$ . An early discussion of smoothing by averaging over regions of variable size, which could grow as long as they did not cross over edges, was given by Roetling [12].<sup>1</sup> This approach should, in principle, yield optimal smoothings, since it averages over the entire region containing each point; but this requires that the picture be explicitly segmented into the regions, which is a computationally costly process. A simpler approach is to use a neighborhood of each point, but to allow each neighborhood to be as large as possible, as long as it does not go outside the region containing that point, i.e., to use the  $N(x, y)$ 's as neighborhoods. An unsuccessful attempt to automatically determine an averaging neighborhood size at each point of a picture was reported in [8].

The results of smoothing the pictures of Fig. 1 using the  $N(x, y)$ 's as averaging neighborhoods are shown in Fig. 4. The smoothing appears to be quite good, especially in (b).

A problem with most methods of edge detection is that they make use of symmetrical edge detection operators, and so cannot

<sup>1</sup> For a general review of region growing techniques, see [13].

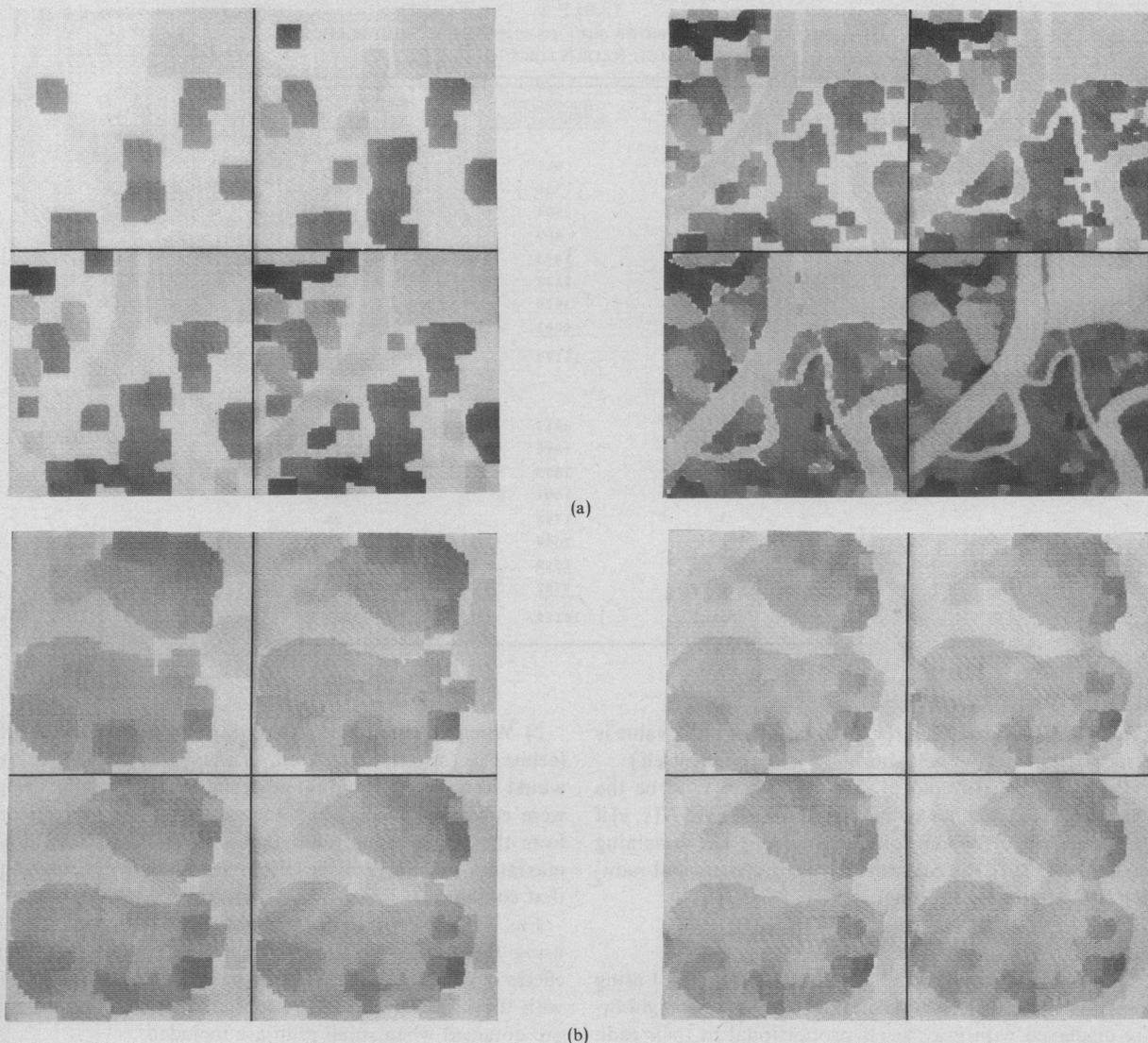


Fig. 3. (a) Steps in reconstructing Fig. 1(a) from its SPAN. Successive parts show radii  $7, 7 + 6, \dots, 7 + 6 + \dots + 0$ . (b) Same for Fig. 1(b).

take full advantage of the uniformity of the regions between which an edge is to be detected, if these regions are of very different sizes.<sup>2</sup> For example, suppose that regions  $A$  and  $B$  have widths  $a$  and  $b$  where  $a < b$ , and that we detect edges by taking differences of averages computed over neighborhoods of size  $c$ . If  $c > a$ , our detector will be too big for region  $A$ , so that parts of the picture lying beyond  $A$  will be included in the  $A$  average. But if  $c \leq a < b$ , our detector sees only part of region  $B$ , and cannot take full advantage of  $B$ 's uniformity. For a discussion of this problem, see [16] and compare [9].

Here again, an optimal approach would be to use the regions themselves as averaging neighborhoods. (Once we have extracted the regions, we know where their edges are; but we still need to do the averaging in order to determine the strengths of these edges.) However, explicitly determining the regions is computationally costly. A simpler idea is to use the maximal  $N(x, y)$ 's as averaging neighborhoods. If  $P$  is a point where two such  $N(x, y)$ 's touch, we can take the difference of averages over the  $N(x, y)$ 's as the edge

value at  $P$ . Note that  $P$  may be interior to a region (e.g., if a region is rectangular, there will be many maximal  $N(x, y)$ 's inside it); but in such a case the difference of averages will be close to zero. On the other hand, if  $P$  is on an edge between two contrasting regions, there should be maximal  $N(x, y)$ 's contained in the two regions and meeting at  $P$ , and the difference of their averages will be high. Thus, the edge values computed in this way should be high along region edges, and low or zero elsewhere. (In the pictures shown below, edge values less than 6 have been suppressed.)

The results of applying this edge detection scheme to the pictures of Fig. 1, using pairs of adjacent, nonoverlapping maximal  $N(x, y)$ 's, are shown in Fig. 5(a), (b). For comparison, the result of applying a simple gradient operator (the "Roberts cross") to the images is shown in Fig. 5(c), (d). The SPAN edges are much stronger than the gradient edges for Fig. 1(b). The noisiness of the SPAN edges for Fig. 1(a) is primarily due to the points for which  $r = 0$ .

## V. DISCUSSION AND CONCLUSIONS

In this paper we have defined a technique for piecewise approximation of pictures, based on maximal neighborhoods, that can be applied to pictures which are approximately piecewise constant.

<sup>2</sup> An exception is the Hueckel edge detector [14], [15], but this is based on a neighborhood of fixed size.

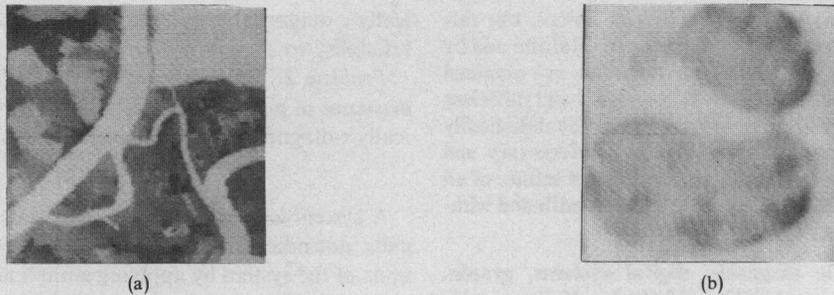


Fig. 4. Results of smoothing Fig. 1(a), (b) using the maximal neighborhood of each point as an averaging neighborhood.

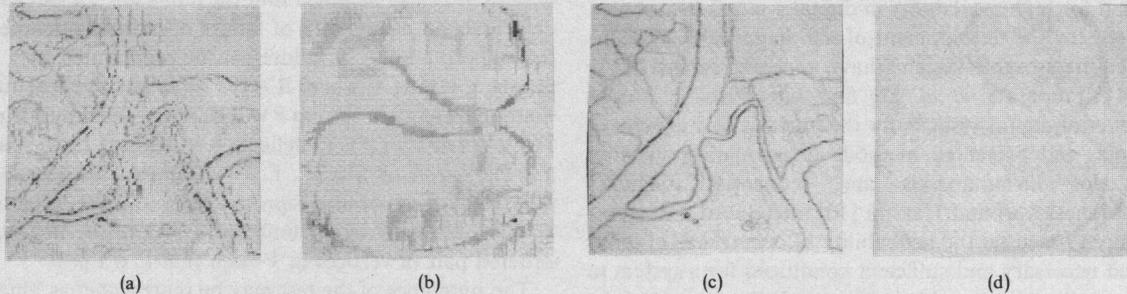


Fig. 5. Edges detected on Fig. 1(a), (b) using pairs of adjacent, nonoverlapping SPAN neighborhoods (a), (b), and using the Roberts cross operator (c), (d).

As the examples show, there are real-world classes of pictures that can be treated as piecewise constant for purposes of SPAN construction. On the other hand, for a picture that contains a significant gray level ramp, the SPAN method breaks down, since it will attempt to approximate the ramp by a staircase.

It should be possible, in principle, to generalize the SPAN approach to a wider class of pictures, e.g., pictures that are approximately piecewise linear, rather than piecewise constant, in gray level. Here, for each neighborhood  $N_r(x, y)$ , we would test the hypothesis that its gray level population is a good fit to a ramp, say in the least squares sense. The largest  $r$  for which this fit is sufficiently good would define the neighborhood  $N(x, y)$ , and we could then find the maximal  $N(x, y)$ 's as above. On region growing in piecewise linear pictures, see [7].

In conclusion, the SPAN approach is a useful generalization of Blum's MAT concept to noisy, unsegmented pictures. Like the MAT, it provides natural, concise approximations to such pictures that can be used for purposes of encoding, recognition, and description, while avoiding the commitment of segmentation. Since it is a parallel method, it could be implemented quite efficiently on a parallel array processor.

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### Some Existence Theorems for Probabilistically Diagnosable Systems

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**Abstract**—This correspondence is concerned with probabilistic fault diagnosis for digital systems. The model considered in this correspondence is the diagnostic model introduced by Maheshwari

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