ESTIMATING MOTION/STRUCTURE FROM LINE CORRESPONDENCES: A ROBUST LINEAR ALGORITHM AND UNIQUENESS THEOREMS

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Abstract

A closed-form solution to motion and structure from line correspondences in monocular perspective image sequences is presented. The algorithm requires a minimum of 13 lines over three perspective views. Redundancy in the data provides overdetermination to combat noise. The estimates can be used as an initial guess for further optimization [8]. A unique solution to motion and structure is guaranteed if and only if the line configuration is not degenerate and the translation between any two views does not vanish. Necessary and sufficient conditions for degenerate spatial line configurations have been derived.

Simulations are performed which show the performance of the algorithm in the presence of noise.

1. INTRODUCTION

With monocular image sequences, the motion parameters and the structure of the scene generally can be derived up to a global scale factor. A feature based approach conventionally involves the following steps: First, the correspondence (or displacement vector) of some features are established. Then, the motion and structure are computed from these correspondences. This paper is mainly devoted to the second step: the computation of motion and structure from line correspondences. An approach is proposed in [7] that combines the motion analysis between consecutive views in a long sequence to reach a higher level understanding of the motion.

The choice of the type of features depends on their availability in the images and the reliability of their measurement. When points are not available in large quantities, other features such as lines or contours can be used. Since the higher level features like lines, edges and contours are determined by a set of pixels, the redundancy in the pixels that form a line make it possible to locate those features accurately.

We discuss in this paper the use of straight lines without known end points, since the end points of a line are not stable with respect to viewpoint. For example, the end points are often occluding points which do not correspond to physical points and they are not fixed in 3-D as the view point changes. Many factors such as lighting and surface reflection often change the position of two ends of a line. However, the location and orientation of the line can generally be determined reliably by a line fitter to edge points.

From line correspondences among three perspective views, iteration is used for solving motion parameters from line correspondences [2], [4], [1]. A convergence to a solution is not guaranteed by those iterative algorithms.

Spetsakis and Aloimonos [5] and Liu and Huang[3] recently proposed linear algorithms for estimating motion and structure parameters from line correspondences. Though Spetsakis and Aloimonos claim a closed form solution, many problems remain to be solved. First, many spurious solutions are generated by their algorithm. The number of spurious solutions is so large that the computation is very inefficient and unreliable. Second, the algorithm fails for some types of motion even though the translation does not vanish. Third, they have not answered the question of uniqueness. By substituting many spurious solutions into the original equations, as proposed by them, is any solution that satisfies the equation correct? Fourth, what is the necessary and sufficient condition for the algorithm to give a unique solution? Finally, since their algorithm is for the exact data, no noise is considered. An algorithm needs to be designed in the presence of noise and the sensitivity of the algorithm to the noise needs to be investigated. Similar criticisms can be made of the algorithm of Liu and Huang. These problems are taken up in this paper.

Because the space is limited, we omit most proofs here. A full length discussion is presented in [9]. The algorithm is derived in Section 2. Section 3 is devoted to the problem of degeneracy and uniqueness. Section 4 presents the simulation results. Section 5 presents concluding remarks.

2. SOLUTION AND ALGORITHM

The goal is to determine the relative motion between the camera and the scene, as well as the structure of the scene. Let the (camera) coordinate system be fixed on the camera with the origin coinciding with the focal point of the camera, and the **Z** axis coinciding with the optical axis and pointing in front. (In the camera coordinate system, the scene is moving.) Since any unit can be used to measure the three-dimensional distance, we choose the focal length of the camera as a unit for simplicity. Thus the focal length is unity, and the image plane is located at $z=1$. Visible objects are always located in front of the image plane, *i.e.*, $z>1$. We assume the scene is rigid.

First we introduce some notation. Matrices are denoted by capital italics. Vectors are denoted by boldface either capital or small. A 3-dimensional column vector is specified by $(s_1, s_2, s_3)^t$. A vector is sometimes regarded as a column matrix. So vector operations such as cross product ($\times$) and matrix operations such as matrix multiplication are applied to 3-dimensional vectors. Matrix operations precede vector operations. 0 denotes a zero vector. A vector with a hat such as $\hat{\mathbf{T}}$ denotes the corresponding unit vector of the original vector. $\mathbf{a}//\mathbf{b}$ if and only if $\mathbf{a}\times\mathbf{b}=0$. For a matrix $A=[a_{ij}]$, $\|A\|$ denotes the Euclidean norm of the matrix. *i.e.*, $\|[a_{ij}]\|^2=\sum_{ij}a_{ij}^2$. We define a mapping $[\cdot]_\times$ from a 3-dimensional vector to a 3 by 3 matrix:

$$[(x_1, x_2, x_3)^t]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{2.1}$$

Using this mapping, we can express cross operation of two vectors by the matrix product of a 3 by 3 matrix and a column matrix:

$$\mathbf{X}\times\mathbf{Y} = [\mathbf{X}]_\times\mathbf{Y} \tag{2.2}$$

When we say that a vector $\mathbf{x}$ or a matrix E is *essentially determined* or *essentially solved*, we mean that it is determined up to a scale factor, i.e., $\mathbf{x}=k\mathbf{y}$ or $E=kE_s$ with known $\mathbf{y}$ or E_s and unknown real number k.

Essential Parameters

In a coordinate system fixed on the camera, a line passing through a point $\mathbf{x}_p$ (to be specific, let $\mathbf{x}_p$ be the point on the line that is the closest to the origin), with direction $\mathbf{l}$ at time t_0, can be expressed in parametrical form:

$$\mathbf{x}_0 = \mathbf{x}_p + k\mathbf{l} \tag{2.3}$$

where the subscript in $\mathbf{x}_0$ means time t_0, and k is a parameter. At another time instant, t_1, the line is moved from t_0 by a rotation represented by a rotation matrix R and then a translation represented by a translation vector $\mathbf{T}$. That is, any point at position $\mathbf{x}_1$ at time t_1 is related to its original position $\mathbf{x}_0$ at times t_0 by

$$\mathbf{x}_1 = R\,\mathbf{x}_0 + \mathbf{T} \tag{2.4}$$

The line equation at time t_1 is

$$t_1: \qquad \mathbf{x}_1 = R\,\mathbf{x}_0 + \mathbf{T} = (R\,\mathbf{x}_p + \mathbf{T}) + kR\,\mathbf{l}. \tag{2.5}$$

Similarly at another time instant t_2, the line is rotated by a rotation matrix S and then translated by a vector $\mathbf{U}$ from time t_0. The line equation at t_2 is

$$t_2: \qquad \mathbf{x}_2 = S\,\mathbf{x}_0 + \mathbf{U} = (S\,\mathbf{x}_p + \mathbf{U}) + kS\,\mathbf{l}. \tag{2.6}$$

The order of the three time instants t_0, t_1 and t_2 can be arbitrary.

We define the *characteristic normal* of a line as the normal of the plane that passes through the line and the focal point of the camera (also the origin). Since the characteristic normal is orthogonal to the line and the position vector of a point on the line, it is easy to get the characteristic normals at the three time instants from (2.4)-(2.6):

$$t_0: \quad \mathbf{n}_0 = \mathbf{x}_p \times \mathbf{l} \tag{2.7}$$

$$t_1: \quad \mathbf{n}_1 = (R\,\mathbf{x}_p + \mathbf{T}) \times R\,\mathbf{l} = R\,((\mathbf{x}_p + R^{-1}\mathbf{T}) \times \mathbf{l}) = R\,(\mathbf{n}_0 + R^{-1}\mathbf{T} \times \mathbf{l}) \tag{2.8}$$

$$t_2: \quad \mathbf{n}_2 = (S\,\mathbf{x}_p + \mathbf{U}) \times S\,\mathbf{l} = S\,((\mathbf{x}_p + S^{-1}\mathbf{U}) \times \mathbf{l}) = S\,(\mathbf{n}_0 + S^{-1}\mathbf{U} \times \mathbf{l}) \tag{2.9}$$

Equation (2.8) gives

$$R^{-1}\mathbf{n}_1 = \mathbf{n}_0 + R^{-1}\mathbf{T} \times \mathbf{l} \tag{2.10}$$

Using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ and (2.10) yields

$$\mathbf{n}_0 \times R^{-1}\mathbf{n}_1 = \mathbf{n}_0 \times (R^{-1}\mathbf{T} \times \mathbf{l}) = (\mathbf{n}_0\cdot\mathbf{l})R^{-1}\mathbf{T} - (\mathbf{n}_0\cdot R^{-1}\mathbf{T})\mathbf{l} = -(\mathbf{n}_0\cdot R^{-1}\mathbf{T})\mathbf{l} \tag{2.11}$$

The last equation follows from the fact that $\mathbf{n}_0\cdot\mathbf{l}=0$. Equation (2.10) gives

$$\mathbf{n}_0\cdot R^{-1}\mathbf{T} = (R^{-1}\mathbf{n}_1 - R^{-1}\mathbf{T}\times\mathbf{l})\cdot R^{-1}\mathbf{T} = R^{-1}\mathbf{n}_1\cdot R^{-1}\mathbf{T} = \mathbf{n}_1\cdot\mathbf{T} \tag{2.12}$$

Equation (2.11) and (2.12) yield

$$\mathbf{n}_0 \times R^{-1}\mathbf{n}_1 = -(\mathbf{n}_1\cdot\mathbf{T})\mathbf{l} \tag{2.13}$$

Similarly we get

$$\mathbf{n}_0 \times S^{-1}\mathbf{n}_2 = -(\mathbf{n}_2\cdot\mathbf{U})\mathbf{l} \tag{2.14}$$

Multiplying both sides of (2.13) by $\mathbf{n}_2\cdot\mathbf{U}$ and those of (2.14) by $\mathbf{n}_1\cdot\mathbf{T}$ yields

$$(\mathbf{n}_2\cdot\mathbf{U})(\mathbf{n}_0 \times R^{-1}\mathbf{n}_1) = (\mathbf{n}_1\cdot\mathbf{T})(\mathbf{n}_0 \times S^{-1}\mathbf{n}_2) \tag{2.15}$$

Or,

$$[\mathbf{n}_0]_\times \mathbf{B} = 0 \tag{2.16}$$

where $\mathbf{B} = (\mathbf{n}_2\cdot\mathbf{U})R^{-1}\mathbf{n}_1 - (\mathbf{n}_1\cdot\mathbf{T})S^{-1}\mathbf{n}_2$. Letting $R=[\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3]$ and $S=[\mathbf{S}_1\ \mathbf{S}_2\ \mathbf{S}_3]$, $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \begin{bmatrix} \mathbf{n}_1^t(\mathbf{R}_1\mathbf{U}^t - \mathbf{T}\mathbf{S}_1^t)\mathbf{n}_2 \\ \mathbf{n}_1^t(\mathbf{R}_2\mathbf{U}^t - \mathbf{T}\mathbf{S}_2^t)\mathbf{n}_2 \\ \mathbf{n}_1^t(\mathbf{R}_3\mathbf{U}^t - \mathbf{T}\mathbf{S}_3^t)\mathbf{n}_2 \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{n}_1^t E\,\mathbf{n}_2 \\ \mathbf{n}_1^t F\,\mathbf{n}_2 \\ \mathbf{n}_1^t G\,\mathbf{n}_2 \end{bmatrix} \tag{2.17}$$

where define the *intermediate parameters* (E, F, G):

$$E = \mathbf{R}_1\mathbf{U}^t - \mathbf{T}\mathbf{S}_1^t, \qquad F = \mathbf{R}_2\mathbf{U}^t - \mathbf{T}\mathbf{S}_2^t, \qquad G = \mathbf{R}_3\mathbf{U}^t - \mathbf{T}\mathbf{S}_3^t. \tag{2.18}$$

We have

$$[\mathbf{n}_0]_\times \begin{bmatrix} \mathbf{n}_1^t E\,\mathbf{n}_2 \\ \mathbf{n}_1^t F\,\mathbf{n}_2 \\ \mathbf{n}_1^t G\,\mathbf{n}_2 \end{bmatrix} = 0 \tag{2.19}$$

Equation (2.19) is a vector equation involving motion parameters $R, \mathbf{T}, S, \mathbf{U}$ and observables $\mathbf{n}_0$, $\mathbf{n}_1$, and $\mathbf{n}_2$. The nonzero scale factor of $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$ is arbitrary in (2.19). The three scalar equations in (2.19) are linear in the $9\times3=27$ components of the intermediate parameters (E, F, G). Since $rank([\mathbf{n}_0]_\times) = 2$ for $\mathbf{n}_0 \neq 0$, (2.19) has at most two independent scalar equations. From each line correspondence over three perspective views, we get a set of corresponding characteristic normals: $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$. If we have at least 13 line correspondences over three views, we might have 26 independent scalar equations. If so, we can essentially solve for the intermediate parameters (E, F, G) based on (2.19). The condition to have 26 independent scalar equations is discussed in the next section. Now it is assumed that the intermediate parameters (E, F, G) are essentially determined. For convenience, we solve for the normalized intermediate parameters (E_s, F_s, G_s) with $\|E_s\|^2 + \|F_s\|^2 + \|G_s\|^2 = 1$, such that

$$(E_s, F_s, G_s) = \alpha(E, F, G) \tag{2.20}$$

with unknown real number α. The motion parameters are to be determined in terms of the intermediate parameters. $\|\mathbf{T}\|^2 + \|\mathbf{U}\|^2$ cannot be determined from the monocular images. For simplicity of notation, we drop the subscript s and let $\|E\|^2 + \|F\|^2 + \|G\|^2 = 1$, with the understanding that (E, F, G) are known only up to a scale factor.

Motion Parameters from Essential Parameters

Let $\mathbf{V}_i = \mathbf{T}\times\mathbf{R}_i$, $i=1,2,3$. From (2.18) we have $E^t\mathbf{V}_1=0$, $F^t\mathbf{V}_2=0$ and $G^t\mathbf{V}_3=0$. If the ranks of E, F, G are all equal to two, $\mathbf{V}_i$ can be essentially determined from (E, F, G). Then the translation vector $\mathbf{T}$ can be essentially determined by $\mathbf{T}\cdot\mathbf{V}_i=0$, $i=1,2,3$. However the ranks of E, F, G are not always equal to 2. The following theorem enumerates all the possible cases.

Theorem 1. Assume $\mathbf{T}\neq0$ and $\mathbf{U}\neq0$. Then there exist unit vectors $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ such that

$$E^t\mathbf{V}_1 = 0, \tag{2.21}$$

$$F^t\mathbf{V}_2 = 0, \tag{2.22}$$

$$G^t\mathbf{V}_3 = 0. \tag{2.23}$$

and the ranks of E, F and G fall into three cases.

Case 1. All of E, F, G have rank two. $\mathbf{V}_i$ is then essentially determined. Let $A=[\mathbf{V}_1\ \mathbf{V}_2\ \mathbf{V}_3]$. Then $rank(A)=2$ and $\mathbf{T}$ is essentially determined by $A^t\mathbf{T}=0$.

Case 2. Two of E, F, G have rank two, and the third has rank one. Without loss of generality, let $rank(E)=1$. Let $A=[\mathbf{V}_2\ \mathbf{V}_3]$. If $rank(A)=2$, $\mathbf{T}$ is still essentially determined by $A^t\mathbf{T}=0$. Otherwise, $\mathbf{T}$ is essentially determined by $\mathbf{T} // (\mathbf{E}_i\times\mathbf{V}_2)\times\mathbf{V}_2$, where $\mathbf{E}_i$ is any nonzero column vector of E. $(\mathbf{E}_i\times\mathbf{V}_2)\times\mathbf{V}_2\neq0$.

Case 3. Only one of E, F, G has rank two, the other two matrices have rank one. Without loss of generality let $rank(G)=2$. Then there are two orthogonal solutions in (2.21) and (2.22), respectively:

$$E^t\mathbf{V}_{1a}=0, \qquad E^t\mathbf{V}_{1b}=0,$$

$$F^t\mathbf{V}_{2a}=0, \qquad F^t\mathbf{V}_{2b}=0.$$

where $\mathbf{V}_{1a}\cdot\mathbf{V}_{1b}=0$ and $\mathbf{V}_{2a}\cdot\mathbf{V}_{2b}=0$. One and only one of two equations

$$\mathbf{V}_3\cdot(\mathbf{V}_{1a}\times\mathbf{V}_{1b})=0 \tag{2.24}$$

and

$$\mathbf{V}_3\cdot(\mathbf{V}_{2a}\times\mathbf{V}_{2b})=0 \tag{2.25}$$

hold. $\mathbf{T} // \mathbf{V}_{1a}\times\mathbf{V}_{1b}$ if (2.24) holds. $\mathbf{T} // \mathbf{V}_{2a}\times\mathbf{V}_{2b}$ if (2.25) holds. □

From Theorem 1, we know that $\mathbf{T}$ can be essentially determined. Similarly if we apply E^t, F^t, G^t to Theorem 1, we know $\mathbf{U}$ can also be essentially determined. The following theorem states the uniqueness of the solution for motion parameters from the intermediate parameters. The condition $\mathbf{T}\neq 0$, $\mathbf{U}\neq 0$ and $R^t\mathbf{T}\neq S^t\mathbf{U}$ is called *distinct location condition.* In section 5 we will see that this condition turns out to be a necessary condition for unique solution of intermediate parameters from (2.19). It is a sufficient condition in following theorem.

Theorem 2. Given (E, F, G), the solution for R, $\mathbf{T}$, S, $\mathbf{U}$ is unique provided $\mathbf{T}\neq 0$, $\mathbf{U}\neq 0$ and $R^t\mathbf{T}\neq S^t\mathbf{U}$.

Proof. From Theorem 1, we can determine $\hat{\mathbf{T}}_s$ and $\hat{\mathbf{U}}_s$, such that $\mathbf{T}=s_1\|\mathbf{T}\|\hat{\mathbf{T}}_s$ and $\mathbf{U}=s_2\|\mathbf{U}\|\hat{\mathbf{U}}_s$, where $s_1,s_2\in\{-1, 1\}$. For four combinations of the values of s_1 and s_2, we have four sets of equations:

$$(s_1, s_2):\quad \begin{cases} E=s_1\|\mathbf{U}\|\mathbf{R}_1\hat{\mathbf{U}}_s^t-s_2\|\mathbf{T}\|\hat{\mathbf{T}}_s\mathbf{S}_1^t \\ F=s_1\|\mathbf{U}\|\mathbf{R}_2\hat{\mathbf{U}}_s^t-s_2\|\mathbf{T}\|\hat{\mathbf{T}}_s\mathbf{S}_2^t \\ G=s_1\|\mathbf{U}\|\mathbf{R}_3\hat{\mathbf{U}}_s^t-s_2\|\mathbf{T}\|\hat{\mathbf{T}}_s\mathbf{S}_3^t \end{cases} \qquad (2.26).$$

Premultiplying both sides of the first equation in (2.26) yields

$$[\hat{\mathbf{T}}_s]_\times E = s_1\|\mathbf{U}\|[\hat{\mathbf{T}}_s]_\times\mathbf{R}_1\hat{\mathbf{U}}_s^t \qquad (2.27)$$

Post-multiplying both sides by $\hat{\mathbf{U}}_s$ gives

$$[\hat{\mathbf{T}}_s]_\times E\hat{\mathbf{U}}_s = s_1\|\mathbf{U}\|[\hat{\mathbf{T}}_s]_\times\mathbf{R}_1 \qquad (2.28)$$

Applying the same operations to the second and the third equations in (2.26) gives the other two equations similar to (2.28). Combining these three equations yields

$$[\hat{\mathbf{T}}_s]_\times[E\hat{\mathbf{U}}_s \;\; F\hat{\mathbf{U}}_s \;\; G\hat{\mathbf{U}}_s] = s_1\|\mathbf{U}\|[\hat{\mathbf{T}}_s]_\times R \qquad (2.29)$$

Since R is a rotation matrix, $\|R\mathbf{x}\|=\|\mathbf{x}\|$. We get $\|\mathbf{U}\|$ from (2.29):

$$\|\mathbf{U}\| = \|[\hat{\mathbf{T}}_s]_\times\|^{-1}\|[\hat{\mathbf{T}}_s]_\times[E\hat{\mathbf{U}}_s \;\; F\hat{\mathbf{U}}_s \;\; G\hat{\mathbf{U}}_s]\| \qquad (2.30)$$

Considering the transposed version of E, F, G, similarly we have

$$[\hat{\mathbf{U}}_s]_\times[E^t\hat{\mathbf{T}}_s \;\; F^t\hat{\mathbf{T}}_s \;\; G^t\hat{\mathbf{T}}_s] = -s_2\|\mathbf{T}\|[\hat{\mathbf{U}}_s]_\times S \qquad (2.31)$$

$\|\mathbf{T}\|$ is determined similarly to (2.30). The equations (2.29) and (2.31) both have the form $A=BR$. In the presence of noise, we solve for a rotation matrix R for $A=BR$ such that

$$\|A-BR\|=min \quad \textit{subject to}: R \textit{ is a rotation matrix} \qquad (2.32)$$

The solution will be presented later with the algorithm.

However, there exist four combinations for all the possible signs of (s_1, s_2) in (2.26). The following Lemma states that only one combination has a solution for rotation matrices R and S from (2.26).

Lemma 2. Assume $\mathbf{T}\neq 0$, $\mathbf{U}\neq 0$, and $R^t\mathbf{T}\neq S^t\mathbf{U}$. Only one assignment for (s_1, s_2), $s_1, s_2\in\{-1, 1\}$, has a solution for rotation matrices R and S from (2.26).

By substituting into (2.26) the four assignments for (s_1, s_2), we get unique solution R and S for each assignment. The assignment of (s_1, s_2) that satisfies (2.26) gives the solution for motion parameters. □

One the other hand, (E, F, G) can only be essentially determined, i.e., up to a scale factor. From (2.29)-(2.31), it is easy to be seen that the scale factor does not affect the solution of the rotation matrices R and S. However the translation vector pair $(\mathbf{T}, \mathbf{U})$ is only essentially determined. We can choose any sign for (E, F, G) and solve for the translation vector pair to get $\mathbf{T}_s$ and $\mathbf{U}_s$ such that $(\mathbf{T}, \mathbf{U})=\alpha(\mathbf{T}_s, \mathbf{U}_s)$ with unknown α. The absolute value of α can not determined from monocular images. The sign of α can be determined in the following.

Structure, and Sign of Translation Vectors

We can solve R and S and $(\mathbf{T}, \mathbf{U})=\alpha(\mathbf{T}_s, \mathbf{U}_s)$ with unknown α. From (2.7)-(2.9) we get $\mathbf{l}\cdot\mathbf{n}_0=0$, $\mathbf{l}\cdot R^{-1}\mathbf{n}_1=0$ and $\mathbf{l}\cdot S^{-1}\mathbf{n}_2=0$. For each line we solve for a unit vector $\hat{\mathbf{l}}$ such that $\mathbf{l}/\!/\hat{\mathbf{l}}$ by

$$\|[\mathbf{n}_0 \;\; R^{-1}\mathbf{n}_1 \;\; S^{-1}\mathbf{n}_2]^t\hat{\mathbf{l}}\|=min \qquad (2.33)$$

If the rank of $[\mathbf{n}_0 \;\; R^{-1}\mathbf{n}_1 \;\; S^{-1}\mathbf{n}_2]$ is no more than one, the line position cannot be recovered.

For each line let $\mathbf{x}_p$ be a point on the line that is the closest to the origin. $d_0\triangleq\|\mathbf{x}_p\|$ is the positive distance of the line to the origin. Since $\mathbf{x}_p\cdot\mathbf{l}=0$, from (2.7) we have

$$\|\mathbf{n}_0\|=\|\mathbf{x}_p\times\mathbf{l}\|=\|\mathbf{x}_p\|\,\|\mathbf{l}\| \qquad (2.34)$$

Though we use (2.7)-(2.9) to define the characteristic normals, the scale factor of those normals is immaterial since it will be canceled out later in (2.36). Using (2.13) and (2.34) yields

$$|\mathbf{T}\cdot\mathbf{n}_1|=\|\mathbf{l}\|^{-1}\|\mathbf{n}_0\times R^{-1}\mathbf{n}_1\|=\|\mathbf{x}_p\|\,\|\mathbf{n}_0\|^{-1}\|\mathbf{n}_0\times R^{-1}\mathbf{n}_1\| \qquad (2.35)$$

Dividing both sides by $\|\mathbf{n}_1\|$ gives the distance of the line to the origin

$$d_0=\|\mathbf{x}_p\|=\|\hat{\mathbf{n}}_0\times R^{-1}\hat{\mathbf{n}}_1\|^{-1}|\mathbf{T}\cdot\hat{\mathbf{n}}_1| \qquad (2.36)$$

Notice that d_0 is proportional to $\|\mathbf{T}\|$. When $\mathbf{T}_s$ replaces for $\mathbf{T}$ in (2.36), $\|\mathbf{x}_p\|$ is essentially determined.

Let $\hat{\mathbf{v}}$ be a unit vector that is parallel to $\mathbf{x}_p$ and always points to the $\mathbf{Z}$ direction. That is, $\hat{\mathbf{v}}=\pm\|\mathbf{n}_0\times\mathbf{l}\|^{-1}\mathbf{n}_0\times\mathbf{l}$, such that $\mathbf{Z}\cdot\hat{\mathbf{v}}>0$. Then $\mathbf{x}_p=\pm d_0\hat{\mathbf{v}}$.

To determine the sign for the translation vectors, we assume that $\mathbf{x}_p$, the point on the line that is the closest to the origin, has a positive Z component for a majority of lines. This assumption is called *majority positive-z assumption.* We can then determine the sign for translation vectors [9]. The motion from t_0 to t_2 can be analyzed in a similar way.

In the Presence of Noise

Since short lines in the images are generally not as reliably determined as long lines, less weight should be assigned to the short lines in (2.19) when solving E, F, G. Let the length of the lines in the images at time t_i be l_i, $i=0, 1, 2$. The weight for the line is

$$(l_0^{-1}+l_1^{-1}+l_2^{-1})^{-1} \qquad (2.42)$$

A simple way to ensure that two independent equations are always included is including all three equations of (2.19). These three equations are scaled by the weight in (2.42) in the system of linear equations with (2.19) for all the lines.

In the presence of noise, a noise corrupted matrix is generally of full rank, the conditions on the rank of the matrices should then be modified for the algorithm. A discussion of the sensitivity of the eigenvectors to the perturbation of the matrix can be found in [6]. A rough measurement for the error of the eigenvector associated with the smallest eigenvalue λ_1 is $(\lambda_1-\lambda_2)^{-1}$ where λ_2 is the second smallest eigenvalue. The solution of $\mathbf{V}_i$ $i=1, 2, 3$ in (2.21)-(2.23) is the eigenvector of EE^t, FF^t, GG^t, respectively, associated with the smallest eigenvalue. The reliability of those solutions is roughly proportional to the difference of the smallest eigenvalue and the second smallest one. Case 1 and Case 2 in Theorem 1 can be implemented by combining them together using a weighted A. Let the *ith* column vectors of A be weighted by the difference of the two smallest eigenvalues of EE^t, FF^t and GG^t, for $i=1, 2, 3$, respectively. For example, if $rank(E)$ is close to one, the corresponding weight in A is close to zero, which is Case 2.

We have observed considerable improvements in simulations by using the weighted schemes discussed above.

In determining the distance d_0, the motion from t_0 to t_1 and that from t_0 to t_2 can both be used to enchance the robustness.

Algorithm

In the algorithm, ε denotes a small positive threshold to accommodate noise. Without noise, ε=0. With noise ε can be estimated by the approach in [6] or determined empirically. Though ε should be different in different parts of the algorithm, a single ε will be used for the simplicity of notation.

(i) *solving for* (E, F, G) *up to a scale factor.*

Given n line correspondences over three views. For each line let the unit characteristic normal at time t_i be $\mathbf{n}_i$, i=0, 1, 2. Solve for (E, F, G) such that

$$\sum_{lines} weight \left\| [\mathbf{n}_0]_\times \begin{bmatrix} \mathbf{n}_1^t E \mathbf{n}_2 \\ \mathbf{n}_1^t F \mathbf{n}_2 \\ \mathbf{n}_1^t G \mathbf{n}_2 \end{bmatrix} \right\|^2 = min \quad \text{(A.1)}$$

with $\|E\|^2+\|F\|^2+\|G\|^2=1$, where the weight for each line is given in (2.42). (A.1) can be written in the form

$$D\mathbf{y}=0 \quad \text{(A.2)}$$

where D is a $3n$ by 27 matrix determined by characteristic normals, and $\mathbf{y}$ is a 27-dimensional unit vector. The solution for unit vector $\mathbf{y}$ is the unit eigenvector of D^tD associated with the smallest eigenvalue.

(ii) *Determining unit vectors* $\hat{\mathbf{T}}_s$ *and* $\hat{\mathbf{U}}_s$ *such that* $\mathbf{T}//\hat{\mathbf{T}}_s$ *and* $\mathbf{U}//\hat{\mathbf{U}}_s$.

Let $H_e \triangleq [\mathbf{h}_{e1}\ \mathbf{h}_{e2}\ \mathbf{h}_{e3}]$, $F_f \triangleq [\mathbf{h}_{f1}\ \mathbf{h}_{f2}\ \mathbf{h}_{f3}]$ and $H_g \triangleq [\mathbf{h}_{g1}\ \mathbf{h}_{g2}\ \mathbf{h}_{g3}]$ be orthogonal matrices such that

$$H_e^t E E^t H_e = diag(\lambda_{e1}, \lambda_{e2}, \lambda_{e3}), \quad \lambda_{e1} \le \lambda_{e2} \le \lambda_{e3}; \quad \text{(A.3)}$$

$$H_f^t F F^t H_f = diag(\lambda_{f1}, \lambda_{f2}, \lambda_{f3}), \quad \lambda_{f1} \le \lambda_{f2} \le \lambda_{f3}; \quad \text{(A.4)}$$

$$H_g^t G G^t H_g = diag(\lambda_{g1}, \lambda_{g2}, \lambda_{g3}), \quad \lambda_{g1} \le \lambda_{g2} \le \lambda_{g3}. \quad \text{(A.5)}$$

Case 1. The medium of the set $C=\{\lambda_{e2}, \lambda_{f2}, \lambda_{g2}\}$ is larger than ε. Let $A=[(\lambda_{e2}-\lambda_{e1})\mathbf{h}_{e1}\ (\lambda_{f2}-\lambda_{f1})\mathbf{h}_{f1}\ (\lambda_{g2}-\lambda_{g1})\mathbf{h}_{g1}]$. a) If the second smallest eigenvalue of A^tA is larger than ε ($rank(A) \ge 2$), $\hat{\mathbf{T}}_s$ is determined up to a scale factor by

$$\|A^t \hat{\mathbf{T}}_s\| = min \quad \text{(A.6)}$$

b) Otherwise ($rank(A)$=1 numerically), determine the smallest number in set C. If λ_{e2} is the smallest in set C, then $\hat{\mathbf{T}}_s // (\mathbf{E}_i \times \mathbf{h}_{f1}) \times \mathbf{h}_{f1}$, where $\mathbf{E}_i$ is a nonzero column vector of E. If λ_{f2} or λ_{g2} is the smallest in C, $\hat{\mathbf{T}}_s$ is determined by the similar equation (circularly rotating e, f, g and E, F, G).

Case 2. The medium of the set $C=\{\lambda_{e2}, \lambda_{f2}, \lambda_{g2}\}$ is not larger than ε. Determine the maximum of the set C. Without loss of generality, assume λ_{g2} is the maximum.

$$\hat{\mathbf{T}}_s = \begin{cases} \mathbf{h}_{e1} \times \mathbf{h}_{e2} & if\ |\mathbf{h}_{g1} \cdot (\mathbf{h}_{e1} \times \mathbf{h}_{e2})| < |\mathbf{h}_{g1} \cdot (\mathbf{h}_{f1} \times \mathbf{h}_{f2})| \\ \mathbf{h}_{f1} \times \mathbf{h}_{f2} & otherwise \end{cases} \quad \text{(A.7)}$$

Replacing E, F, G by E^t, F^t, G^t, similarly determine $\hat{\mathbf{U}}_s$.

(iii). *Determining* R *and* S.

Let

$$G_R = [\hat{\mathbf{T}}_s]_\times [E\hat{\mathbf{U}}_s\ \ F\hat{\mathbf{U}}_s\ \ G\hat{\mathbf{U}}_s]$$

$$G_S = -[\hat{\mathbf{U}}_s]_\times [E^t\hat{\mathbf{T}}_s\ \ F^t\hat{\mathbf{T}}_s\ \ G^t\hat{\mathbf{T}}_s] \quad \text{(A.8)}$$

and $\|\mathbf{U}\| = \|G_R\|/\sqrt{2}$, $\|\mathbf{T}\| = \|G_S\|/\sqrt{2}$. Then let $G_R \leftarrow \|\mathbf{U}\|^{-1} G_R$, and $G_S \leftarrow \|\mathbf{T}\|^{-1} G_S$. Solve for the following rotation matrices R_p, R_n, S_p and S_n such that

$$\|G_R - [\hat{\mathbf{T}}_s]_\times R_p\| = min \qquad \|-G_R - [\hat{\mathbf{T}}_s]_\times R_n\| = min \quad \text{(A.9)}$$

$$\|G_S - [\hat{\mathbf{U}}_s]_\times S_p\| = min \qquad \|-G_S - [\hat{\mathbf{U}}_s]_\times S_n\| = min \quad \text{(A.10)}$$

Both (A.9) and (A.10) have the form (noticing $\|D-CR\| = \|R^tC^t - D^t\|$):

$$\|RC-D\| = \min \quad subject\ to: R\ is\ a\ rotation\ matrix \quad \text{(A.11)}$$

Where $C=[\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3]$, $D=[\mathbf{D}_1\ \mathbf{D}_2\ \mathbf{D}_3]$. The solution of (A.11) is as follows.

Define a 4 by 4 matrix B by

$$B = \sum_{i=1}^{3} B_i^t B_i \quad \text{(A.12)}$$

where

$$B_i = \begin{bmatrix} 0 & (\mathbf{C}_i - \mathbf{D}_i)^t \\ \mathbf{D}_i - \mathbf{C}_i & [\mathbf{D}_i + \mathbf{C}_i]_\times \end{bmatrix} \quad \text{(A.13)}$$

Let $\mathbf{q}=(q_0, q_1, q_2, q_3)^t$ be the unit eigenvector of B associated with the smallest eigenvalue. The solution of rotation matrix R in (A.11) is

$$R = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_2q_1+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_3q_1-q_0q_2) & 2(q_3q_2+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \quad \text{(A.14)}$$

Substituting four assignments

$$(1, 1, R_p, S_p), \quad (-1, 1, R_n, S_p),$$
$$(1, 1, R_p, S_n), \quad (-1, -1, R_n, S_n)$$

for (s_t, s_u, R, S) in

$$\begin{cases} E = s_t \|\mathbf{U}\| \mathbf{R}_1 \hat{\mathbf{U}}_s^t - s_u \|\mathbf{T}\| \hat{\mathbf{T}}_s \mathbf{S}_1^t \\ F = s_t \|\mathbf{U}\| \mathbf{R}_2 \hat{\mathbf{U}}_s^t - s_u \|\mathbf{T}\| \hat{\mathbf{T}}_s \mathbf{S}_2^t \\ G = s_t \|\mathbf{U}\| \mathbf{R}_2 \hat{\mathbf{U}}_s^t - s_u \|\mathbf{T}\| \hat{\mathbf{T}}_s \mathbf{S}_3^t \end{cases} \quad \text{(A.15)}$$

the set best satisfying (A.15), in the sense of Euclidean norm, gives the assignment of (s_t, s_u, R, S). Then

$$\mathbf{T} = s_t \|\mathbf{T}\| \hat{\mathbf{T}}_s, \qquad \mathbf{U} = s_u \|\mathbf{U}\| \hat{\mathbf{U}}_s. \quad \text{(A.16)}$$

(iv). *Determining structure* $\hat{\mathbf{l}}$ *and* $\mathbf{x}_p$.

For each line, solve for the direction of the line represented by a unit vector $\hat{\mathbf{l}}$, such that

$$\|[\mathbf{n}_0\ R^{-1}\mathbf{n}_1\ S^{-1}\mathbf{n}_2]^t \hat{\mathbf{l}}\| = min \quad \text{(A.17)}$$

$$d_0 = \frac{|\mathbf{T} \cdot \hat{\mathbf{n}}_1|}{2\|\hat{\mathbf{n}}_0 \times R^{-1}\hat{\mathbf{n}}_1\|} + \frac{|\mathbf{U} \cdot \hat{\mathbf{n}}_2|}{2\|\hat{\mathbf{n}}_0 \times S^{-1}\hat{\mathbf{n}}_2\|} \quad \text{(A.18)}$$

$$\hat{\mathbf{v}} = \pm \frac{\hat{\mathbf{n}}_0 \times \hat{\mathbf{l}}}{\|\hat{\mathbf{n}}_0 \times \hat{\mathbf{l}}\|} \quad \text{(A.19)}$$

where the sign is such that the third component of $\mathbf{v}$ is non-negative.

Let POS and NEG be empty sets. For each line i do the following: If

$$|\mathbf{n}_1 \cdot (d_0 R\, \hat{\mathbf{v}} + \mathbf{T})| + |\mathbf{n}_2 \cdot (d_0 S\, \hat{\mathbf{v}} + \mathbf{U})| < |\mathbf{n}_1 \cdot (d_0 R\, \hat{\mathbf{v}} - \mathbf{T})| + |\mathbf{n}_2 \cdot (d_0 S\, \hat{\mathbf{v}} - \mathbf{U})|$$

i is added into the set POS. Otherwise, i is added to the set NEG.

Finally, if $\|POS\| > \|NEG\|$, for each line i,

$$\mathbf{x}_p = \begin{cases} d_0 \hat{\mathbf{v}} & if\ i \in POS \\ -d_0 \hat{\mathbf{v}} & otherwise \end{cases} \quad \text{(A.20)}$$

Otherwise if $\|POS\| < \|NEG\|$,

$$\mathbf{T} \leftarrow -\mathbf{T}, \qquad \mathbf{U} \leftarrow -\mathbf{U}. \quad \text{(A.21)}$$

For each line i,

$$\mathbf{x}_p = \begin{cases} -d_0 \hat{\mathbf{v}} & if\ i \in POS \\ d_0 \hat{\mathbf{v}} & otherwise \end{cases} \quad \text{(A.22)}$$

3. DEGENERACY

From just two views, it is impossible to determine motion and structure from line correspondences: An image line with the focal point determines a 3-D plane. Two images with the focal point determine two 3-D planes whose intersection gives the 3-D line. We can move, slightly and arbitrarily, one image with its focal point and the new 3-D plane still intersect the corresponding 3-D plane and so, determines a new 3-D line. The same is true for other lines. In other words, motion parameters can be arbitrary and the corresponding 3-D structure of lines exists such that they yield the same pair of images. However, intersection of three planes generally is not a line. This is why three views are generally enough to determine motion and structure.

In the last section, it is established by Theorem 2 that as long as $\mathbf{T}\neq 0$, $\mathbf{U}\neq 0$ and $\mathbf{T}^t R \neq \mathbf{U}^t S$ the solution of motion parameters from the intermediate parameters (E, F, G) is unique.

First, let us see what the condition

$$\mathbf{T}\neq 0, \qquad \mathbf{U}\neq 0, \qquad \mathbf{T}^t R \neq \mathbf{U}^t S \tag{3.1}$$

means. Let a world coordinate system be fixed with the scene and coincide with the camera coordinate system (that we have used earlier) at time t_0. It can be easily shown that the position of the focal point at time t_1, in the world coordinate system, is at $\mathbf{O}_1=-R^t\mathbf{T}$. Similarly the position of the focal point at time t_2 is at $\mathbf{O}_2=-S^t\mathbf{U}$. Thus the condition in (3.1) is equivalent to the condition

$$\mathbf{O}_1\neq 0, \qquad \mathbf{O}_2\neq 0, \qquad \mathbf{O}_1\neq\mathbf{O}_2 \tag{3.2}$$

That is to say that any two positions of the focal point of the moving camera do not coincide, or in other words, the translation between any two views does not vanish. So the conditions in (3.1) and (3.2) are called *distinct location condition.*

The following theorem gives the necessary and sufficient conditions for degeneracy in terms of 3-D line configurations at time t_0 and the motion parameters.

Theorem 3. (E, F, G) is not essentially determined by (2.19) or equivalently, $rank(D)<26$ in (A.2), if and only if there exist no trivial parameters $(\tilde{E}, \tilde{F}, \tilde{G})$ such that

$$[\mathbf{n}]_\times \begin{bmatrix} ((\mathbf{x}_p-\mathbf{O}_1)\times\mathbf{l})^t \tilde{E}((\mathbf{x}_p-\mathbf{O}_2)\times\mathbf{l}) \\ ((\mathbf{x}_p-\mathbf{O}_1)\times\mathbf{l})^t \tilde{F}((\mathbf{x}_p-\mathbf{O}_2)\times\mathbf{l}) \\ ((\mathbf{x}_p-\mathbf{O}_1)\times\mathbf{l})^t \tilde{G}((\mathbf{x}_p-\mathbf{O}_2)\times\mathbf{l}) \end{bmatrix} = 0 \tag{3.3}$$

is satisfied for all lines $\mathbf{x}=\mathbf{x}_p+k\mathbf{l}$ at time t_0. $(\tilde{E}, \tilde{F}, \tilde{G})$ is trivial if and only if

$$(\tilde{E}, \tilde{F}, \tilde{G})=\alpha(P_1-Q_1^t, P_2-Q_2^t, P_3-Q_3^t) \tag{3.4}$$

for some real number α. where P_i and Q_i are matrices with the $i-th$ column being $\mathbf{O}_1$ and $\mathbf{O}_2$, respectively, and the other columns are zero vectors. □

Corollary. If the distinct location condition is not satisfied, the intermediate parameters is not essentially determined by (2.19). □

Another form of necessary and sufficient condition is presented in the following theorem.

Theorem 4. Assume 13 line correspondences at time t_0, with the $i-th$ line being represented by $\mathbf{x}=\mathbf{x}_{pi}+\mathbf{l}_i$ ($\mathbf{x}_{pi}$ is the point closest to the origin) and the normal for the plane that passes though the line and $\mathbf{O}_k$ is $\tilde{\mathbf{n}}_{ki}$, $k=1,2$, $i=1, 2 \cdots 13$. Then (E, F, G) is not essentially determined by (2.19), or equivalently, $rank(D)<26$ in (A.2), if and only if there exist no a_i, b_i, $i=1, 2, 3, \cdots, 13$, not all of which are zeros, such that

$$\sum_{i=1}^{13}(a_i\mathbf{x}_{pi}+b_i\mathbf{l}_i)\tilde{\mathbf{n}}_{2i}\tilde{\mathbf{n}}_{3i}=0 \tag{3.5}$$

where **ab** denotes tensor out product of **a** and **b**. For m-dimensional vector **a** and n-dimensional vector **b**, **ab** has mn components which are products of all the possible combinations between an element in **a** and an element in **b**. □

From Theorem 2 though theorem 4 and the majority positive-z assumption, we come to the conclusion that if the distinct location condition is satisfied and the line structure is not degenerate (see (3.3) or (3.5)), the motion parameters and the structure of the line can be uniquely determined.

4. SIMULATIONS

Simulations have been conducted to check the correctness of the algorithm and the sensitivity of the solution to the noise.

The lines are generated randomly for time t_0. In noise-free cases, the error in the solution are of the order of 10^{-10} because of computer round off errors.

The visible end points of the lines at each time are projected onto the image. These image coordinates are digitized according to the resolution of the images to simulate noise. The images have a size 2 by 2. For a 512 by 512 image, the image coordinates have 2×512 evenly spaced levels for u and v coordinates, respectively, accounting for a line fitting process for real images. The image coordinates are digitized to the closest levels. For each line, two digitized images of the two visible end points and focal point of the camera (the origin) determine the characteristic normal of this line. In the presence of noise (real image coordinates are digitized according to the image resolution), the errors depend on the configuration of the lines randomly generated. To show the general sensitivity to the noise, the average errors over 100 random sequences (randomly generated lines at time t_0) are recorded. Fig. 1 shows a sample sequence.

The errors shown in the following are all relative errors. The relative error of a matrix (or vector) is defined by the Euclidean norm of the difference between the estimated and the true matrix (or vector) divided by the Euclidean norm of the true matrix (or vector). Fig. 2 through Fig. 4 show the relative errors for the number of lines ranging from 13 to 30. The image resolution is 512 by 512. The motion parameters are as follows: R corresponds to a rotation about an axis (1, 1, 1) by an angle 6°, and S to a rotation about (0, 1, −1) by 5°. $\mathbf{T}$=(1, −1, 3). $\mathbf{U}$=(1, 1, −2.5). It can be seen from those figures that, with a minimal 13 lines, the algorithm does not give reliable estimates of motion parameters with the 512 by 512 resolution. Some short lines in the images may give very unreliable characteristic normals in the presence of noise. Degenerate configurations are more likely to be generated with fewer lines. When the number of line correspondences increases, the errors decreases considerably. When the number of line correspondences is around 20, we get reasonably accurate results. Fig. 4 shows the average errors of the recovered direction of lines $\mathbf{l}$ (solid lines) and those of recovered relative errors of the distance of the line to the origin, $\|\mathbf{x}_p\|$, (dashed lines).

5. CONCLUSIONS

A new linear algorithm is presented for determining the motion parameters and the structure of the lines. The algorithm uses a minimal of 13 lines but more lines are needed to obtain more accurate results in the presence of noise. The algorithm is complete and the uniqueness of the solution is proved. The degenerate conditions for the structure of the lines are presented. Derivation for a more intuitive condition is suggested for future research. The errors of this algorithm can be estimated using the approach in [6].

The estimates of this algorithm can be used as an initial guess for an iterative process that optimizes the solution based on a good objective function (See [8] for a two-step approach). Since a better objective function is used and a good initial guess is available, solutions can be significantly improved through iterations.

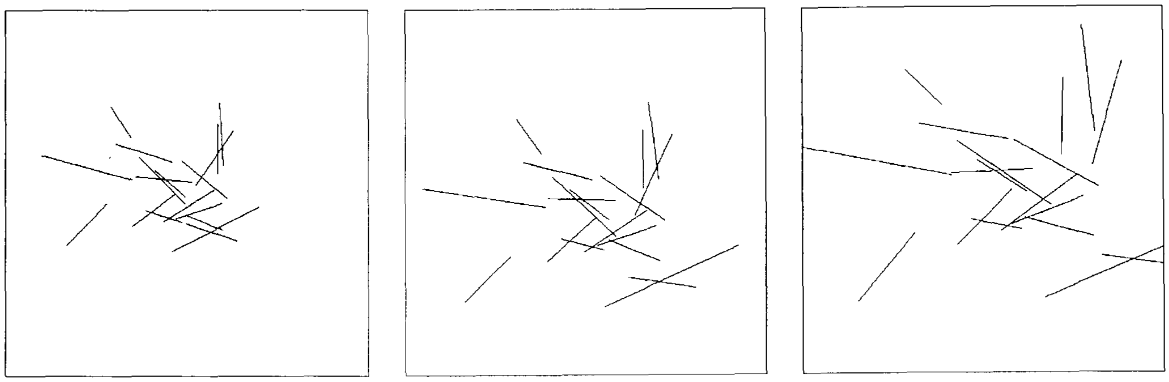

Fig.1. A sample sequence with 18 lines

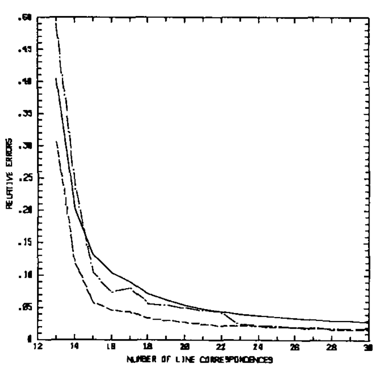

Fig.2. Relative Errors of (E, F, G) (solid lines), R (dashed lines) and $\mathbf{T}$ (dot-dashed lines) versus number of line correspondences.

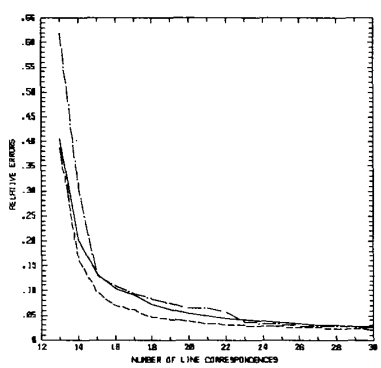

Fig.3. Relative Errors of (E, F, G) (solid lines), S (dashed lines) and $\mathbf{U}$ (dot-dashed lines) versus number of line correspondences.

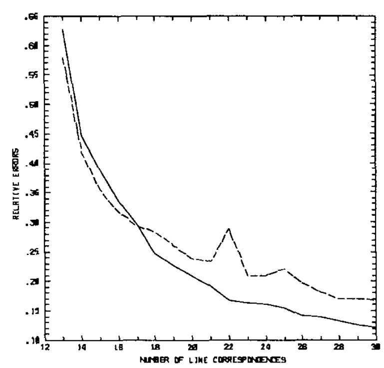

Fig.4. Average Relative Errors of line direction $\hat{\mathbf{l}}$ (solid lines), and line distance $\|\mathbf{x}_p\|$ (dashed lines) versus number of line correspondences.

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