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SUFFICIENT CONDITIONS FOR DOUBLE OR UNIQUE SOLUTION OF MOTION AND STRUCTURE

Xiaoping Hu and *Narendra Ahuja*,

Beckman Inst. and Dept. of ECE, Univ. of Illinois at Urbana-Champaign
405 N. Mathews, Urbana, IL 61801, USA

email: xh, ahuja@uirvld.csl.uiuc.edu

ABSTRACT

This paper presents several sufficient conditions for double or unique solution of the problem of motion and structure from two monocular images. We show that: *1*. 5 correspondences of points that do not lie on two lines in the image plane suffice to determine a pure rotation uniquely; *2*. 6 correspondences of points that do not lie on two lines in the image plane and do not correspond to space points lying on a specific quadric surface suffice to determine a motion with nonzero translation uniquely; *3*. each Maybank quadric can sustain at most two physically acceptable motion solutions and surface interpretations, provided that a sufficient number of correspondences are present; *4*. in the plane motion case, 6 correspondences of points that do not lie on a quadratic curve in the image plane will only admit the true motion and structure and their duals as solutions. We list several properties of the essential matrix $\mathbf{T}\times\mathbf{R}$ and the plane motion matrix $\mathbf{R}+\mathbf{TN}^T$, both of which are frequently used in the motion and structure estimation problem.

1. Introduction

This paper concerns the uniqueness of solution of general motion parameters of a rigid surface from two monocular views. This problem can be stated as follows: with how many correspondences of image points and under what conditions can we have a unique solution for $\mathbf{R}$ and a solution up to a scalar for $\mathbf{T}$ from the motion equation

$$Z'_i[\,x'_i\; y'_i\; 1\,]^T = Z_i\mathbf{R}[\,x_i\; y_i\; 1\,]^T + \mathbf{T}\;,\; i=1,2,\cdots,n, \quad (1\text{-}1)$$

where (x'_i, y'_i) and (x_i, y_i) are a pair of image point correspondences, $\mathbf{R}$ a rotation matrix, $\mathbf{T}$ a vector, and Z'_i and Z_i any positive constants?

Under the assumption that a sufficient number of correspondences is provided, Longuet-Higgins ([9]) and Negahdaripour ([21]) analyzed the problem of estimating surface structure. Longuet-Higgins obtained the following results: *1*. If one interpretation of a pair of photographs locates the visible texture elements in a plane, then so does every other. *2*. Otherwise, if the images are ambiguous, every interpretation will locate the visible elements on a special type of quadric surface, called Maybank Quadric; but in that case the pair of images cannot sustain more than three distinct and physically acceptable interpretations. Negahdaripour further concluded that: *1*. In the case of hyperboloids of one sheet and hyperbolic paraboloids, th ere can be three possible solutions. 2. In the case of circular cylinders and intersecting planes, there are at most two solutions.

But how many correspondences are sufficient and what surface conditions should they satisfy to yield unique or a finite number of solutions?

The only existing sufficient condition for unique solution of general surface motion is having 8 correspondences of points that do not lie on a quadric surface passing through the origin of the coordinate system and $-\mathbf{R}_0^T\mathbf{T}_0$, where $\mathbf{R}_0$ and $\mathbf{T}_0$ define the true motion ([8][1][17]). Although for a planar surface, we have some sufficient conditions for determining the motion uniquely ([10][8][12][17][18]), so far we do not know if and under what conditions the motion of a plane admits a solution leading to non-planar surface.

The goal of this paper is to present the new and less stringent sufficient conditions for the uniqueness problem that are summarized in the abstract.

2. Preliminary Results

Our main results will be based on the following preliminary results whose proofs are omitted due to lack of space.

We will use Equation (1-1) to represent motion. We use $\mathbf{\Theta}$, $\mathbf{\Theta}'$ to represent vectors $[\,x\;y\;1]^T, [\,x'\;y'\;1\,]^T$ so that $\mathbf{X}=Z\mathbf{\Theta}$, $\mathbf{X}'=Z'\mathbf{\Theta}'$; we call $\mathbf{\Theta}$ as well as (x,y) a point in the image plane.

We will frequently use the following *rigidity* property ([6][17]) of a rotation matrix: for any two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$,

$$\mathbf{R}\;(\mathbf{X}_1\times\mathbf{X}_2) = (\mathbf{R}\mathbf{X}_1)\times(\mathbf{R}\mathbf{X}_2)\,, \quad (2\text{-}1)$$

since it is equivalent to the orthonormality and unit determinant property ([17][18]).

Our discussion in this paper will be based on the the so-called *essential matrix* ([15])

$$\mathbf{E}\;\triangleq\;\mathbf{T}\times\mathbf{R}\;=\;\mathbf{GR}, \quad (2\text{-}2)$$

where

$$\mathbf{G}\;=\;\mathbf{T}\times\;=\;\begin{bmatrix}0 & -t_3 & t_2\\ t_3 & 0 & -t_1\\ -t_2 & t_1 & 0\end{bmatrix}\;=\;\begin{bmatrix}\mathbf{g}_1^T\\ \mathbf{g}_2^T\\ \mathbf{g}_3^T\end{bmatrix} \quad (2\text{-}3)$$

is a skew-symmetric matrix. A matrix is called *essential* or *decomposable* if and only if it is decomposable into the form of (2-4) with $\mathbf{T}$ a nonzero vector and $\mathbf{R}$ a rotation matrix.

Lemma 2.1 .

A matrix $\mathbf{E}$ can be represented as $\mathbf{T}\times\mathbf{R}$ with $\mathbf{T}$ a nonzero vector and $\mathbf{R}$ a rotation matrix, if and only if it can be represented as $\mathbf{R}_1(\mathbf{T}_1\times)$ for some rotation matrix $\mathbf{R}_1$ and some nonzero vector $\mathbf{T}_1$. Or, a matrix $\mathbf{E}$ is essential if and only if there exist some rotation matrices $\mathbf{R}_i$, i = 1,2, and a non-zero vector $\mathbf{T}_3$ such that

$$\mathbf{E}\;=\;\mathbf{R}_1(\mathbf{T}_3\times\mathbf{R}_2)\,, \quad (2\text{-}4)$$

where only one of $\mathbf{R}_1$, $\mathbf{T}_3$, or $\mathbf{R}_2$ can be arbitrary at one time. ‖

Corollary 2. 1 .

$\mathbf{E}$ is decomposable, if and only if $\mathbf{RE}$ or $\mathbf{ER}$ is decomposable, where $\mathbf{R}$ is any rotation matrix; $\mathbf{E}$ is decomposable if and only if $\mathbf{E}^T$ is decomposable. ‖

Many necessary and sufficient conditions have been obtained for the essential matrix ([12][15][16][17]); the following condition expressed in terms of the elements of $\mathbf{E}$ may more clearly capture the properties of the essential matrix.

Lemma 2.2

A matrix $\mathbf{E}=[\,\mathbf{e}_1\;\mathbf{e}_2\;\mathbf{e}_3\,]^T$ can be represented as $\mathbf{T}\times\mathbf{R}$, where $\mathbf{T}=[\,t_1\;t_2\;t_3\,]^T\neq 0$ and $\mathbf{R}$ is a rotation matrix, if and

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only if there exist three distinct indices i,j,k among 1,2,3 such that one of the following three situations occurs:

1.

$$\frac{\mathbf{e}_1}{\mathbf{e}_2 \cdot \mathbf{e}_3} + \frac{\mathbf{e}_2}{\mathbf{e}_3 \cdot \mathbf{e}_1} + \frac{\mathbf{e}_3}{\mathbf{e}_1 \cdot \mathbf{e}_2} = 0, \tag{2-5}$$

where

$$\mathbf{e}_m \cdot \mathbf{e}_n \neq 0, \quad \text{for any } m \neq n, \; m,n = 1,2,3. \tag{2-6}$$

In this case,

$$t_1 t_2 t_3 \neq 0. \tag{2-7}$$

2.

$$\|\mathbf{e}_i\| = \|\mathbf{e}_j\| > 0, \quad \|\mathbf{e}_k\| = 0, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0. \tag{2-8}$$

In this case,

$$t_i = t_j = 0, \quad \text{but} \quad t_k \neq 0. \tag{2-9}$$

3.

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, \; \mathbf{e}_j \times \mathbf{e}_k = 0, \; \|\mathbf{e}_i\|^2 = \|\mathbf{e}_j\|^2 + \|\mathbf{e}_k\|^2,$$

$$\|\mathbf{e}_j\| > 0, \; \|\mathbf{e}_k\| > 0. \tag{2-10}$$

In this case,

$$t_i = 0, \; t_j t_k \neq 0. \; \blacksquare \tag{2-11}$$

Longuet-Higgins derived (2-17) (see [16]), which of course is one of the conditions for a matrix to be essential.

The next lemma is a generalized version of Zhuang et al.'s ([1]) and can be proven using Zhuang et al.'s method.

Lemma 2.3 .

Assume an arbitrary matrix $\mathbf{K}$ is invertible. Then $\mathbf{K}^T\mathbf{E}$ is skew-symmetric if and only if $\mathbf{E} = \mathbf{GK}$ for some skew-symmetric matrix $\mathbf{G}$. ∎

Lemma 2.4 .

If $\mathbf{E}$ is an essential matrix then both $\mathbf{E}$ and $\mathbf{E} + \mathbf{E}^T$ are rank reduced. ∎

3. Pure Rotation Case

We present the following theorems without proofs.

Theorem 3.1 . ([17][18])

If the motion is known to be a pure rotation $\mathbf{R}_0$, then 2 correspondences of image points $\Theta_i = [x_i \; y_i \; 1]^T$, $\Theta'_i = [x'_i \; y'_i \; 1]^T$, $i = 1,2$, exclude any spurious pure rotation solutions, and the rotation matrix is uniquely given by

$$\mathbf{R} = \left[\gamma_1 \Theta'_1 \; \gamma_2 \Theta'_2 \; (\gamma_1 \Theta'_1) \times (\gamma_2 \Theta'_2) \right] \left[\Theta_1 \; \Theta_2 \; \Theta_1 \times \Theta_2 \right]^{-1} \tag{3-1}$$

where

$$\gamma_i \overset{\Delta}{=} \frac{Z'_i}{Z_i} = \frac{\|\Theta_i\|}{\|\Theta'_i\|} = \sqrt{\frac{x_i^2 + y_i^2 + 1}{x_i'^2 + y_i'^2 + 1}}, \; i = 1,2. \; \blacksquare \tag{3-2}$$

The above theorem does not exclude spurious motion solutions with nonzero translation. The next theorem deals with this problem.

Theorem 3.2 . (The proof uses the method in Section 4.)

If the motion is a pure rotation (but this condition is not known in advance), then a sufficient condition for determining the motion uniquely is having 5 correspondences of points in the image plane with no 3 of the 5 points colinear. ∎

4. General Motion Case

In this section, we consider general motion where $\mathbf{T} \neq 0$.

The following theorem states a sufficient condition eliminating any spurious solution with zero translation.

Theorem 4.1 .

If the true motion involves a rotation $\mathbf{R}_0$ and a nonzero translation $\mathbf{T}_0$, a sufficient condition for excluding a pure rotation as a solution is having 5 correspondences of points with no 3 of the 5 points colinear in the image plane.

Proof: Let $\Theta' = (x', y', 1)^T$ and $\Theta = (x, y, 1)^T$ be a correspondence pair in the image plane. Then we should have

$$\Theta'^T(\mathbf{T}_0 \times \mathbf{R}_0)\Theta = 0. \tag{4-1}$$

Now assume a pure rotation $\mathbf{R}$ yields the same correspondence pair, then we should also have

$$\gamma\Theta' = \mathbf{R}\Theta, \tag{4-2}$$

where γ is similarly defined as γ_i is in (3-3). Replacing (4-2) into (4-1) gives

$$\Theta^T\mathbf{R}^T(\mathbf{T}_0 \times \mathbf{R}_0)\Theta = 0. \tag{4-3}$$

(4-3) indicates that the points must lie on a quadratic curve. Thus a sufficient condition for excluding a pure rotation as a solution is having 5 correspondences of points with no 3 of the 5 points colinear in the image plane. Q.E.D.

The next theorem states a sufficient condition eliminating any spurious solution with nonzero translation. However, we need to present two lemmas dealing with two particular spurious solutions: in one only the translation is distinct; in the other only the rotation matrix is distinct.

Lemma 4.1 . (Proof is omitted.)

If the true motion is given by $\mathbf{R}_0$ and $\mathbf{T}_0 \neq 0$, then 3 correspondences of points that are non-colinear in the image plane suffice to exclude any spurious solution with the rotation $\mathbf{R}_0$ and a translation $\mathbf{T}$ not parallel to $\mathbf{T}_0$. ∎

Lemma 4.2 .

If the true motion is given by $\mathbf{R}_0$ and $\mathbf{T}_0$, a sufficient condition for excluding any spurious solution with a rotation $\mathbf{R} \neq \mathbf{R}_0$ and a translation $\mathbf{T}$ parallel to $\mathbf{T}_0$, is having 5 correspondences of points with no 3 of the 5 points being colinear in the image plane.

Proof: Given a space point correspondence $\mathbf{X}' = Z'\Theta'$ and $\mathbf{X} = Z\Theta$, we have

$$\mathbf{X}' = \mathbf{R}_0\mathbf{X} + \mathbf{T}_0. \tag{4-4}$$

Although we only know Θ' and Θ, Z' and Z are uniquely determined by (4-4) unless Θ' is parallel to $\mathbf{T}_0$. In any case, for the true depths Z' and Z, equation (4-4) is always satisfied. Now assume we have a spurious motion solution $\mathbf{R}$ and $\mathbf{T} \neq 0$. Then the motion epipolar line equation

$$\Theta'^T\mathbf{E}\Theta = 0 \tag{4-5}$$

must be satisfied, where $\mathbf{E} = \mathbf{T} \times \mathbf{R}$. Multiplying the left side of (4-5) by $Z'Z$ yields

$$\mathbf{X}'^T\mathbf{E}\mathbf{X} = 0. \tag{4-6}$$

Replacing (4-4) into (4-6) we get

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{E})\mathbf{X} + \mathbf{T}_0^T\mathbf{E}\mathbf{X} = 0. \tag{4-7}$$

(4-7) is called *Maybank Quadric* ([28]). With $\mathbf{T} \times \mathbf{T}_0 = 0$, (4-7) reduces to

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{T} \times \mathbf{R})\mathbf{X} = \Theta^T(\mathbf{R}_0^T\mathbf{T}_0 \times \mathbf{R})\Theta = \Theta^T\mathbf{A}\Theta = 0, \tag{4-8}$$

where

$$\mathbf{C} = \mathbf{E}_1 + \mathbf{E}_1^T, \text{with } \mathbf{E}_1 = \mathbf{R}_0^T\mathbf{T} \times \mathbf{R}. \tag{4-9}$$

(4-8) defines a quadratic curve in the image plane. Lemmas 2.1 and 2.4 state that $\mathbf{A}$ has a rank of at most 2. From the planar quadratic curve theory ([20]) we know that if $\det(\mathbf{C}) = 0$, the quadratic curve described by (4-8) can only be a line, or two parallel or intersecting lines, or a point. Given 5 points, if no three of them are colinear in the image plane, then there exists no quadratic curve of the type (4-8) passing through all 5 points. We thus have the lemma. Q.E.D.

Finally, let us consider the general situation where $\mathbf{R} \neq \mathbf{R}_0$ and $\mathbf{T} \times \mathbf{T}_0 \neq 0$. Let us rearrange (4-7) as

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{E} + \mathbf{E}^T\mathbf{R}_0)\mathbf{X} + 2(\mathbf{R}_0^T\mathbf{T}_0)^T(\mathbf{R}_0^T\mathbf{E})\mathbf{X} = 0. \tag{4-10}$$

It has been shown ([1]) that if and only if $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{T} \times \mathbf{T}_0 = 0$, (4-10) will become trivial. Therefore (4-10) implies that spurious solutions arise only when the image points used for correspondences correspond to space points on a quadric surface. We then have the following theorem.

Theorem 4.2 .

If the true motion is $\mathbf{R}_0$ and $\mathbf{T}_0$, then a sufficient condition for excluding any spurious solution $\mathbf{R}$ and $\mathbf{T}$ such that $\mathbf{R} \neq \mathbf{R}_0$ and $\mathbf{T} \neq \mathbf{T}_0$ is having 6 correspondences of points that do not lie on a quadric surface of the type (4-10).

Proof: We first examine the surface shapes that can be represented by (4-10). First of all, the quadric must pass through the origin and $-\mathbf{R}_0^T\mathbf{T}_0$. Lemma 2.1 and Lemma 2.4 indicate that $\mathbf{R}_0^T\mathbf{E}$ is an essential matrix and therefore $\mathbf{R}_0^T\mathbf{E} + \mathbf{E}^T\mathbf{R}_0$ has a zero eigenvalue. From quadric surface theory ([20]) , we know that (4-10) can only represent an elliptic cylinder, a line, a hyperbolic cylinder, a parabolic cylinder, or two parallel or overlapping planes. There exist other quadrics, such as elliptic sphere, one or two sheet hyperboloid, and cone, which cannot be represented by (4-10).

Given a quadric surface

$$\mathbf{X}^T(\mathbf{A} + \mathbf{A}^T)\mathbf{X} + 2\mathbf{B}^T\mathbf{X} = 0\,, \tag{4-11}$$

to express it in the form of (4-9), $\mathbf{B}$ must be dependent on $\mathbf{A}$ and $\mathbf{A}$ must be an essential matrix. That is, we must have some $\mathbf{T}$ and $\mathbf{R}$ such that

$$\mathbf{A} = \mathbf{T}\times\mathbf{R}\ ,\ \ \mathbf{B}^T = (\mathbf{R}_0^T\mathbf{T}_0)^T\mathbf{A}\ . \tag{4-12}$$

Therefore, only the elements of $\mathbf{A}$ can be free variables. But $\mathbf{A}$ can have only 5 free elements because the largest element of $\mathbf{A}$ can be normalized to unity and $\mathbf{A}$ must satisfy (2-5), (2-8), or (2-10) to be an essential matrix. It is then possible that 5 points define a finite number of surfaces of the type (4-11). It follows that 6 points may exclude the possibility that they lie on any quadric surface of the type (4-11), e.g., when the 6 points together with the coordinate origin and $-\mathbf{R}_0^T\mathbf{T}_0$ define an elliptic sphere or a hyperboloid. Q.E.D.

Summarizing the discussion in this section and the last section, we have the following theorem.

Theorem 4.3 .

6 correspondences of image points that do not lie on two lines in the image plane and do not correspond to space points lying on a Maybank Quadric suffice to determine a motion uniquely. ||

We will now discuss the following problem: if the Maybank Quadric is uniquely defined, how many spurious solutions can the surface sustain? We will show that at most one spurious solution can be sustained.

Assume $\mathbf{R}_0, \mathbf{T}_0$ represent the true motion, and $\mathbf{R}_1, \mathbf{T}_1$ and $\mathbf{R}_2, \mathbf{T}_2$, are two sets of spurious solutions such that $\mathbf{R}_i \neq \mathbf{R}_j$, $\mathbf{T}_i \times \mathbf{T}_j \neq 0$, for any $i \neq j$, $i,j \in \{0,1,2\}$. Other situations have been discussed in Lemma 4.1 and Lemma 4.2. Then for a given correspondence Θ' and Θ in the image plane, we should have

$$Z'\Theta' = Z\mathbf{R}_0\Theta + \mathbf{T}_0, \tag{4-13}$$

where Z' and Z are the true depths for Θ' and Θ respectively. Similarly, for the two spurious solutions, there must exist positive numbers D', D, ρ', and ρ such that

$$D'\Theta' = D\mathbf{R}_1\Theta + \mathbf{T}_1\,. \tag{4-14}$$

$$\rho'\Theta' = \rho\mathbf{R}_2\Theta + \mathbf{T}_2\,. \tag{4-15}$$

From (4-14) and (4-15), we have the following motion epipolar line equations:

$$\Theta'^T(\mathbf{T}_1\times\mathbf{R}_1)\Theta \ \stackrel{\Delta}{=}\ \Theta'^T(\mathbf{E}_1)\Theta = 0\,, \tag{4-16}$$

and

$$\Theta'^T(\mathbf{T}_2\times\mathbf{R}_2)\Theta \ \stackrel{\Delta}{=}\ \Theta'^T(\mathbf{E}_2)\Theta = 0\,. \tag{4-17}$$

Now multiplying equations (4-16) and (4-17) by the true depths Z' and substituting $Z'\Theta$ by (4-13), we get

$$Z\Theta^T(\mathbf{R}_0^T\mathbf{E}_1)\Theta + \mathbf{T}_0^T\mathbf{E}_1\Theta = 0\,, \tag{4-18}$$

$$Z\Theta^T(\mathbf{R}_0^T\mathbf{E}_2)\Theta + \mathbf{T}_0^T\mathbf{E}_2\Theta = 0\,. \tag{4-19}$$

Each of (4-18) and (4-19) defines a Maybank Quadric. They must give the same depth for any correspondence except one, as the depths Z' and Z for all correspondences except one are uniquely determined by (4-13). The exception occurs when $\Theta'\times\mathbf{T}_0 = 0$. But in this case, the translation is uniquely determined, and Lemma 4.2 shows that 5 correspondences of points with no 3 of the 5 points being colinear in the image plane uniquely determine the rotation. Therefore the motion in the exceptional case can be uniquely determined. In the discussion below we assume such exceptions do not occur for all motion solutions and correspondence data. That is, $\mathbf{T}_i\times\Theta' \neq 0$ for any available correspondence Θ' and any $\mathbf{T}_i$, $i = 0,1,2$.

Since (4-18) and (4-19) must give the same depth for any correspondence, they must be identical. Thus we must have

$$\mathbf{T}_0^T\mathbf{E}_1 = \mathbf{T}_0^T\mathbf{E}_2\,, \tag{4-20}$$

and

$$\mathbf{R}_0^T\mathbf{E}_1 + \mathbf{E}_1^T\mathbf{R}_0 = \mathbf{R}_0^T\mathbf{E}_2 + \mathbf{E}_2^T\mathbf{R}_0\,. \tag{4-21}$$

Rewrite (4-21) we get

$$\mathbf{R}_0^T(\mathbf{E}_1 - \mathbf{E}_2) = -(\mathbf{E}_1 - \mathbf{E}_2)^T\mathbf{R}_0\,. \tag{4-22}$$

The above equation shows that $\mathbf{R}_0^T(\mathbf{E}_1 - \mathbf{E}_2)$ must be skew-symmetric. Lemma 2.3 states that there must exist some vector $\mathbf{T}_3$ such that

$$\mathbf{E}_1 - \mathbf{E}_2 = \mathbf{T}_3\times\mathbf{R}_0\,. \tag{4-23}$$

Then from (4-20) we know that $\mathbf{T}_0$ must be parallel to $\mathbf{T}_3$ since

$$\mathbf{T}_0^T(\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{T}_0^T(\mathbf{T}_3\times\mathbf{R}_0) = 0\,. \tag{4-24}$$

Therefore we can assume that $\mathbf{T}_3 = \alpha_0\mathbf{T}_0$. Then (4-23) gives

$$\mathbf{E}_1 - \mathbf{E}_2 = \alpha_0\mathbf{T}_0\times\mathbf{R}_0 = \alpha_0\mathbf{E}_0\,. \tag{4-25}$$

For the same reason, for $\mathbf{R}_1$ and $\mathbf{T}_1$ to constitute a valid motion, we must have the following equations

$$D\Theta^T(\mathbf{R}_1^T\mathbf{E}_0)\Theta + \mathbf{T}_1^T\mathbf{E}_0 = 0\,, \tag{4-26}$$

$$D\Theta^T(\mathbf{R}_1^T\mathbf{E}_2)\Theta + \mathbf{T}_1^T\mathbf{E}_2 = 0\,. \tag{4-27}$$

From (4-26) and (4-27) we have

$$\mathbf{E}_2 - \mathbf{E}_0 = \alpha_1\mathbf{E}_1 \tag{4-28}$$

for some constant α_1. Similarly

$$\mathbf{E}_0 - \mathbf{E}_1 = \alpha_2\mathbf{E}_2\,. \tag{4-29}$$

For all three solutions to be acceptable simultaneously, (4-25), (4-28), and (4-29) must hold at the same time. If any constant α_i, $i = 0,1,2$ is zero, then one of the solution must be identical to another, contradicting the assumption that the three solutions are distinct. So in the following we assume $\alpha_i \neq 0$, $i = 0,1,2$. Adding (4-25), (4-28), and (4-29) gives

$$0 = \alpha_0\mathbf{E}_0 + \alpha_1\mathbf{E}_1 + \alpha_2\mathbf{E}_2\,, \tag{4-30}$$

Comparing (4-30) with (4-25) we have·

$$(\alpha_1 + 1)\mathbf{E}_1 = (1 - \alpha_2)\mathbf{E}_2 = 0. \tag{4-31}$$

It is well known that an essential matrix can have at most two decompositions of the form $\mathbf{V}\times\mathbf{U}$ with $\mathbf{V}$ a nonzero vector and $\mathbf{U}$ a rotation matrix (see also Appendix). And if $\mathbf{V}_1\times\mathbf{U}_1$ and $\mathbf{V}_2\times\mathbf{U}_2$ are two distinct decompositions of an essential matrix then we must have $\mathbf{V}_1 = -\mathbf{V}_2$. Since $\mathbf{E}_1$ and $\mathbf{E}_2$ are all essential matrices and $\mathbf{T}_1\times\mathbf{T}_2 \neq 0$ by assumption, we then must have

$$\alpha_1 = -1,\ \ \alpha_2 = 1\,. \tag{4-32}$$

But comparing (4-30) with (4-28) we also have

$$\alpha_0 = 1,\ \alpha_2 = -1\,, \tag{4-33}$$

which contradicts (4-32). Therefore, there exist no three motion solutions which are mutually compatible. As the true motion solution must always hold, therefore there is at most one other solution which gives an alternate interpretation of the surface and the motion.

As 6 points together with the origin and $-\mathbf{R}_0^T\mathbf{T}_0$ can uniquely define a quadric surface, to summarize the discussion

above, we have the following theorem.
Theorem 4.4 .

If 6 correspondences of image points correspond to space points that together with the origin and $-\mathbf{R}_0^T\mathbf{T}_0$ uniquely define a quadric surface and the image points do not lie on two lines in one or both views, then the 6 correspondences suffice to determine the motion parameters to within 2 sets if the quadric surface is a Maybank Quadric, or determine the motion uniquely if the quadric surface is not a Maybank Quadric. ∎

5. Planar Surface Case

It has been shown ([1][10][12][18]) that when the points are coplanar in space, there generally exists a dual motion which always accompanies the true motion and cannot be removed with rigidity constraint; uniqueness is guaranteed unless the dual motion is identical with the true motion or yields negative depths for some visible points. Even worse, there exists an uncertain situation where an infinite number of motion solutions arises, although this situation occurs with zero probability ([9][17]). Longuet-Higgins has proven ([9]) that a planar surface only admits spurious motion solutions leading to planar surfaces. But how many correspondences are needed and what conditions should they satisfy to exclude other solutions leading to non-planar surfaces?

Assume a plane in the space has an equation

$$\mathbf{N}_0^T\mathbf{X} = 1 , \quad [5\text{-}1]$$

and is subject to a motion $\mathbf{R}_0$ and $\mathbf{T}_0$. [5-1] is the non-degeneracy condition of the projection of a plane ([18]). Only when the projection of a plane is not degenerate in both image planes is it possible to find point correspondences. Then it is well known that an image point correspondence pair Θ' and Θ are interrelated by

$$\gamma\Theta' = (\mathbf{R}_0 + \mathbf{T}_0\mathbf{N}_0^T)\Theta \triangleq \mathbf{K}\Theta , \quad [5\text{-}2]$$

where

$$\gamma = \sqrt{\frac{\Theta^T\mathbf{K}^T\mathbf{K}\Theta}{\Theta'^T\Theta'}} , \quad [5\text{-}3]$$

and

$$\mathbf{K} = \mathbf{R}_0 + \mathbf{T}_0\mathbf{N}_0^T . \quad [5\text{-}4]$$

A matrix $\mathbf{K}$ of the type [5-4] is called a *plane motion matrix*, and $\mathbf{R}_0$, $\mathbf{T}_0$, and $\mathbf{N}_0$ are called a motion decomposition of $\mathbf{K}$. It has been shown (Hu [18], Tsai [3]) that unless $\mathbf{R}^T\mathbf{T}$ is parallel to $\mathbf{N}$, the plane motion matrix in [5-4] will still have another decomposition $\mathbf{R}_d$, $\mathbf{T}_d$ and $\mathbf{N}_d$, called dual solution, with $\mathbf{N}_d \times \mathbf{N}_0 \neq 0$. Therefore, as we have shown in [18], the dual solution cannot be removed by the motion model and rigidity constraint, although it can sometimes be identified using positive depth constraint ([10]). There exists an uncertain situation where $\mathbf{K}^T\mathbf{K} = \mathbf{I}$ and $\det(\mathbf{K}) = -1$. In this case, $\mathbf{K}$ has an infinite number of plane motion decompositions ([18]). For this to occur, either the object is transparent and rotated by a half revolution, or one of the image is taken through a mirror. The following theorem is true.
Theorem 5.1 . (Proof is omitted due to lack of space.)

If the points used for correspondences are coplanar and the uncertain situation defined by $\mathbf{K}^T\mathbf{K} = \mathbf{I}$ and $\det(\mathbf{K}) = -1$ does not occur, then 6 correspondences of image points that do not lie on a quadratic curve in the image plane suffice to exclude all spurious motion solutions other than the true solution and the dual solution. ∎

6. Summary

Our main results here can be summarized as follows:
1. When the motion is a pure rotation, 5 correspondences of points with no 3 of the 5 points being colinear in the image plane suffice to determine the motion uniquely.
2. When the motion involves a translation, 6 correspondences of image points that do not lie on two lines in the image plane and do not correspond to space points lying on a Maybank Quadric suffice to determine the motion uniquely.
3. When the points all lie on a plane and the uncertain situation characterized by $\mathbf{K}^T\mathbf{K} = \mathbf{I}$ and $\det(\mathbf{K}) = -1$ does not occur, 6 correspondences of points that do not lie on a quadratic curve in the image plane suffice to restrict the motion to the true and the dual plane motion solutions.
4. Each Maybank quadric can sustain at most two physically acceptable solutions. Therefore, if a Maybank Quadric surface is uniquely defined by 5 or more space points, and the projections of the space points in the image plane do not lie on two lines, then the motion and the surface can be determined to within two solutions.

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