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Motion Estimation under Orthographic Projection

Xiaoping Hu and Narendra Ahuja

Abstract—**In this short paper, we present some new results for the problem of motion estimation under orthographic projection. We refine some basic results obtained by previous researchers and provide more detailed and precise results. We show that, in the two-view problem, when the rotation is around the optical axis, the motion (but not the structure) is uniquely determined. We show that, in the three-view problem, only under certain conditions are the motion and structure uniquely determined. We show that, for any motion problem, if two-view matching cannot determine the motion, only under certain conditions can three- or multiview matching help.**

Index Terms—**Depth, motion, motion estimation, orthography, rigidity, structure.**

I. Introduction

In the literature, two projection models of image formation have been widely used: perspective projection and orthographic projection. The motion estimation problem has been investigated mainly for perspective projection [9]-[18] with some work on orthographic projection [1]-[7]. Ullman [2] started the research on the motion problem with orthographic projection. But in later work, the primary interest of motion researchers has been in perspective projection. This is probably due to the fact that perspective projection models the imaging process of ordinary cameras more accurately and is better conditioned in the sense of determinedness from correspondence data. But, when a long-focus telephoto lens is used, the imaging process can be approximated by orthographic projection if the motion and the object size in the direction of the optical axis are negligible compared with the object distance, although a scale constant may be involved [4]. In medical imaging such as X-ray imaging, the imaging process can be considered as an orthographic projection. Therefore, it is necessary to investigate the motion problem under orthographic projection. Another projection model that lies between the perspective and orthographic projections also has been investigated and is called *paraperspective projection* [8]. In this work, we will discuss the motion estimation problem under orthographic projection only.

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This short paper concerns conditions under which motion is uniquely determined from only monocular image point correspondence data. For the two-view problem, determinedness means that the rotation matrix $\mathbf{R}$ is uniquely determined and the translation vector $\mathbf{T} = [t_1\ t_2\ t_3]^T$ is determined up to a scale and a constant, i.e., $[t_1\ t_2]^T$ is determined up to a scale, and t_3 is not determined. For the three-view problem, determinedness means that all rotation matrices are uniquely determined, the translation vector between the second and the third views is determined to a scale, and the translation vector between the first and the second views is determined to a scale and a constant (similar to the two-view case).

Ullman [2] showed that the two-view motion problem is generally not determined, but the three-view motion problem can generally be determined with four correspondences of projections of noncoplanar space points, and he proposed a nonlinear algorithm for motion estimation from three-view matching. Later, Aloimonos and Brown [3] showed that the two orthographic views of four noncoplanar points admit only four interpretations of the structure of the four points and that it is possible to uniquely recover structure from three orthographic views of three points in space, contradicting Ullman's results. Huang and Lee [1] proposed a linear algorithm for three-view motion estimation. They gave formal proof that the two-view motion problem is generally not determined and the three-view motion problem is generally determined.

In this short paper, we will reexamine some of the problematic results obtained in the above referenced papers. We concern motion estimation only and do not discuss structure estimation. We show that, for monocular vision, the two-view motion is determined if and only if the rotation is around the optical axis, and three-view motion is determined if and only if certain necessary and sufficient conditions are satisfied. We show that, given a sequence of images under orthographic projection, only under certain conditions can the motion between each pair of views be determined by multiview matching. These results contrast those obtained by Ullman [2] and Huang and Lee [1].

In Section II, we present some preliminary results for motion estimation. In Section III, we investigate the two-view motion problem. We show that rotation is uniquely determined if and only if it is around the optical axis. In Section IV, we reexamine Huang and Lee's three-view algorithm and show that the three-view motion problem is determined only under certain conditions. Section V summarizes the paper.

II. Representation of Two-View Motion

We use $x - y$ to denote image coordinates and $X - Y - Z$ to denote real-world coordinates. An image point (x, y) represents the projection of a scene point $\mathbf{X} = (X, Y, Z)$. For orthographic projection, we have

$$\begin{aligned} x &= X \\ y &= Y. \end{aligned} \tag{1}$$

Throughout this work, we will use the following notation. Bold capital letters represent vectors or matrices, italic capital letters coordinates in the space, italic lowercase letters coordinates in the image plane or elements of vectors or matrices. (x, y), (x_i, y_i), and (x_i', y_i') correspond to the projections of $\mathbf{X} = (X, Y, Z)$, $\mathbf{X}_i = (X_i, Y_i, Z_i)$, and $\mathbf{X}_i' = (X_i', Y_i', Z_i')$, respectively. $\mathbf{\Theta}$, $\mathbf{\Theta}_i$, and $\mathbf{\Theta}_i'$ denote $[x, y]^T$, $[x_i, y_i]^T$, and $[x_i', y_i']^T$, respectively.

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Coordinates with prime correspond to the coordinates without prime, e.g., (X, Y, Z) corresponds to (X', Y', Z') in the space and (x, y) to (x', y') in the image plane.

It is well known that $\mathbf{X}$ and $\mathbf{X}'$ can be related by a perfect mathematical model

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \mathbf{R}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \mathbf{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad (2)$$

where $\mathbf{R}$ is the rotation matrix and $\mathbf{T}$ is the translation vector. $\mathbf{R}$ can either be represented by the axis-angle form [12] or by

$$\mathbf{R} = \mathbf{A}_X(\theta_X)\mathbf{A}_Y(\theta_Y)\mathbf{A}_Z(\theta_Z) \quad (3)$$

where $\mathbf{A}_X, \mathbf{A}_Y$, and $\mathbf{A}_Z$ represent rotation matrices around the X, Y, Z axes separately.

By rearranging (2), we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} r_{13} \\ r_{23} \end{bmatrix} Z + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}. \quad (4)$$

By taking inner product of $[r_{23} \ -r_{13}]$ with both sides of (4), we get the resulting product equation

$$x'r_{23} - y'r_{13} = x(r_{23}r_{11} - r_{13}r_{21}) + y(r_{23}r_{12} - r_{13}r_{22}) + r_{23}t_1 - r_{13}t_2. \quad (5)$$

From the identities

$$r_{31} = r_{12}r_{23} - r_{22}r_{13}$$

$$r_{32} = r_{21}r_{13} - r_{11}r_{23} \quad (6)$$

(5) becomes

$$x'r_{23} - y'r_{13} + xr_{32} - yr_{31} = r_{23}t_1 - r_{13}t_2. \quad (7)$$

We shall call (7) the *motion epipolar line equation* because it states that if r_{13} and r_{23} are not zero at the same time, then the correspondence (x', y') of (x, y) lies on a line in the image plane. In fact, a line in the first image frame having equation

$$xr_{32} - yr_{31} = c \quad (8)$$

will correspond to a line in the second image frame having equation

$$x'r_{23} - y'r_{13} + c = r_{23}t_1 - r_{13}t_2 \quad (9)$$

where c is a constant. Note that because

$$r_{13}^2 + r_{23}^2 = r_{31}^2 + r_{32}^2 = 1 - r_{33}^2 \quad (10)$$

$r_{13} = r_{23} = 0$ means that $r_{31} = r_{32} = 0$, $r_{33} = 1$ and hence $\mathbf{R} = \mathbf{A}_Z(\theta_Z)$ for some rotation angle θ_Z. In this case the rotation axis is just the optical axis Z. We thus have the following fact:

Fact 1:

1) If the rotation is not around the optical axis of the camera, then the correspondence (x', y') of a point (x, y) lies on a motion epipolar line defined by the motion parameters via (7).
2) If $r_{13} = r_{23} = 0$, then the correspondence (x', y') of a point (x, y) is uniquely determined by the motion parameters from

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & -\sin\theta_Z \\ \sin\theta_Z & \cos\theta_Z \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}. \quad (11)$$

From (11), we can immediately see that if the rotation is around the optical axis, then the depths are not related to the coordinates in the image plane; hence, the depth and the translation along the depth direction will never be recovered from the correspondences in monocular vision, similar to $\mathbf{T} = 0$ in the perspective perception case.

III. The Two-View Problem

In the two-view problem, we will distinguish rotations around the optical axis from those not around the optical axis and consider the pure rotation case only since a general motion can be reduced to a pure rotation.

A. Rotation around the Optical Axis

In this subsection we consider the case of rotation around the optical axis (including the case where no motion occurs). When the rotation is around the optical axis, the rotation matrix $\mathbf{R}$ can be represented by $\mathbf{A}_Z$ for some θ_Z, that is, $\mathbf{R} = \mathbf{A}_Z$. From (11) and (2) with $t_1 = t_2 = 0$ we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \triangleq \mathbf{R}_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & -\sin\theta_Z \\ \sin\theta_Z & \cos\theta_Z \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \quad (12)$$

Thus, we immediately have the following theorem:

Theorem 1: In the pure rotation case, if the rotation is known to be around the optical axis, then one correspondence of a point not lying on the optical center suffices to determine the rotation uniquely.

Proof: From (12) we have

$$x'(x\sin\theta_Z + y\cos\theta_Z) = y'(x\cos\theta_Z - y\sin\theta_Z). \quad (13)$$

Dividing both sides of the above equation by $\cos\theta_Z$ and rearranging it, we have

$$(x'x + y'y)tg\theta_Z = y'x - x'y \quad (14)$$

or we get two solutions of θ_Z:

$$\theta_Z = \pm tg^{-1}\left(\frac{y'x - x'y}{x'x + y'y}\right). \quad (15)$$

However, only one solution satisfies (12). Thus, we can eliminate the spurious solution using (12) to get the unique solution of θ_Z and hence of $\mathbf{R}$. Solution (15) fails if and only if $x = y = 0$. ■

In practice, if one has many correspondences, one can get an optimal solution of θ_Z from (13) by minimizing some object function. In this case, the rotation matrix is uniquely determined. However, the depths do not relate to the correspondences and hence cannot be recovered.

In Theorem 1, we assume that we already know the rotation is around the optical axis. Later, we will develop a technique that determines if the rotation is really around the optical axis using the correspondence data so that the algorithm here can be applied without heuristics about motion.

B. Rotation around Axis other than the Optical Axis

The other, more general case occurs when the rotation is around some axis other than the optical axis. This case has been dealt with by Huang and Lee [1]. In this case, at least one of r_{13}, r_{23} and one of r_{31}, r_{32} are not zero and the motion epipolar line equation (7) becomes [1]

$$x_i'r_{23} - y_i'r_{13} + x_ir_{32} - y_ir_{31} = 0. \quad (16)$$

Here, we present an alternative solution of (16) and two complementary equations of (16). The new solution is motivated by the fact that the four rotation parameters in (16) are not independent. Therefore, the resulting estimates of the four parameters from

Huang and Lee's linear solution may not be mutually compatible when the correspondence data are noisy.

Since r_{13} and r_{23} cannot be zero at the same time, we will assume, without loss of generality, that $r_{13} \neq 0$ in the rest of this subsection, and the conclusions generally apply to the case where $r_{23} \neq 0$ if only the related equations are properly modified.

When $r_{13} \neq 0$, it is obvious that at least three correspondences are needed to get a linear and unique solution for r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} from

$$x_i' \frac{r_{23}}{r_{13}} + x_i \frac{r_{32}}{r_{13}} - y_i \frac{r_{31}}{r_{13}} = y_i'. \tag{17}$$

Equation (17) is still not the best equation for solution of r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13}, as only two of them can be independent variables, indicated by (10) or the following:

$$1 + \left(\frac{r_{23}}{r_{13}}\right)^2 = \left(\frac{r_{31}}{r_{13}}\right)^2 + \left(\frac{r_{32}}{r_{13}}\right)^2. \tag{18}$$

Therefore, there are only two independent variables in (17). It is thus possible to have a finite number of solutions from two correspondences. Given two correspondences $p_i = (x_i, y_i)$, $p_i' = (x_i', y_i')$, $i = 1, 2$, if there exists no nonzero constant α such that

$$(x_2', y_2', x_2, y_2) = \alpha(x_1', y_1', x_1, y_1). \tag{19}$$

Then we can represent r_{23}/r_{13} and r_{31}/r_{13} as linear functions of r_{32}/r_{13} from (17) and thus determine r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} to within two sets of solutions using (18). We will not go any further in this direction as it is not our main interest. But, this result states that in most cases (i.e., when (19) is not satisfied) two point correspondences suffice to determine r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} to within two sets of solutions. A sufficient condition for determining r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} uniquely is having three correspondences of image points that do not correspond to points colinear in space (in the case of general motion, this condition corresponds to having four correspondences of image points that do not correspond to points coplanar in space [1], [2]).

Equation (18) is a necessary condition of rigidity. If one uses many correspondences to get a linear least squares solution of r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} from (17) they may not necessarily satisfy (18), and hence the solution may not represent a valid motion. So instead, we go the following way. First eliminate r_{23}/r_{13} from any two correspondences to get

$$\beta_{ij} \frac{r_{32}}{r_{13}} + \alpha_{ij} \frac{r_{31}}{r_{13}} = \xi_{ij}, \quad i \neq j \tag{20}$$

where

$$\alpha_{ij} = (y_i x_j' - y_j x_i')$$
$$\beta_{ij} = (x_j x_i' - x_i x_j')$$
$$\xi_{ij} = x_i' y_j' - y_i' x_j'. \tag{21}$$

At least three correspondences are needed to solve for r_{32}/r_{13} and r_{31}/r_{13} from (20). After r_{32}/r_{13} and r_{31}/r_{13} are known, r_{23}/r_{13} can be first determined to within a sign, and then the sign can be determined with the help of one or more correspondences using (17). We list the procedures formally as follows. Let

$$\frac{r_{23}}{r_{13}} \overset{\Delta}{=} a$$
$$\frac{r_{32}}{r_{13}} \overset{\Delta}{=} b$$
$$\frac{r_{31}}{r_{13}} \overset{\Delta}{=} c. \tag{22}$$

Then first solve for b and c from (20) with many correspondences in the sense of linear least squares, and then solve for a from

$$a = \pm\sqrt{b^2 + c^2 - 1} = \pm a_0. \tag{23}$$

To determine whether a_0 or $-a_0$ is the true solution, we calculate

$$d_1 = \sum_i |y_i' - x_i b + y_i c - x_i' a_0| \tag{24}$$

and

$$d_2 = \sum_i |y_i' - x_i b + y_i c + x_i' a_0|. \tag{25}$$

If $d_1 < d_2$, then $a = a_0$; otherwise, $a = -a_0$. When $r_{23} \neq 0$ and $r_{13} = 0$, d_1 and d_2 should be modified.

The rotation parameters can also be solved for from other equations. For this purpose, we present two equations complementary to (17) (see the appendix):

$$\alpha_{ij} r_{11} + \beta_{ij} r_{12} + \gamma_{ij} r_{21} + \eta_{ij} r_{22} + \xi_{ij} r_{33} = \zeta_{ij} \tag{26}$$

$$\eta_{ij} r_{11} - \gamma_{ij} r_{12} - \beta_{ij} r_{21} + \alpha_{ij} r_{22} - \zeta_{ij} r_{33} = -\xi_{ij} \tag{27}$$

where

$$\gamma_{ij} \overset{\Delta}{=} y_j' y_i - y_i' y_j$$
$$\eta_{ij} \overset{\Delta}{=} y_i' x_j - y_j' x_i$$
$$\zeta_{ij} \overset{\Delta}{=} y_i x_j - x_i y_j \tag{28}$$

and α_{ij}, β_{ij}, ξ_{ij} are the same as in (21).

That is, the elements of $\mathbf{R}$ not appearing in (17) are all present in (26) and (27). It appears that if we can find more than five independent equations from (26) or (27) using two-view correspondences, then all the rotation parameters in (26) or (27) can be solved for with two-view correspondences. However, as is shown in the appendix, it turns out that (26) and (27) are trivial once r_{23}/r_{13}, r_{32}/r_{13}, and r_{31}/r_{13} have been determined, and there are never five independent equations obtainable from (26) and (27) using two-view correspondences. So (26) and (27) are redundant equations. But the rotation parameters can be solved for up to a constant from either (17), (26), or (27). A solution of rotation parameters from (26) or (27) is generally more complicated than from (16). Therefore, we will not present such a solution here.

C. Determining Rotation around the Optical Axis from Correspondences

Both Ullman [2] and Huang and Lee [1] do not seem to have realized that there exists a case in which the two-view motion can be uniquely determined without any *a priori* knowledge. Aloimonos and Brown proved that two orthographic views of four noncoplanar points admit only four interpretations of the structure of the four points; however, this conclusion also does not hold in general. Thus, neither of the above results gives an accurate understanding of motion interpretation from images obtained under orthographic projection. We now develop a technique to determine if a rotation is around the optical axis and then show that the rotation can be uniquely determined from correspondence data if that occurs.

When rotation is around the optical axis, we can get at most two independent equations from (17). This property can be used to determine if such a rotation occurs. To show this, we rearrange (17) to get

$$[r_{23}, -r_{13}, r_{32}, -r_{31}] \begin{bmatrix} \Theta_i' \\ \Theta_i \end{bmatrix} = 0. \tag{29}$$

Suppose we have $n(\geq 3)$ noncolinear correspondences (to ensure the noncolinearity of the points in space, it suffices to ensure the noncolinearity of the correspondences in the image plane) and let

$$\mathbf{D}_n \triangleq \begin{bmatrix} \Theta_1' & \Theta_2' & \cdots & \Theta_n' \\ \Theta_1 & \Theta_2 & \cdots & \Theta_n \end{bmatrix}. \tag{30}$$

It has been known [1] that when the rotation is not around the optical axis, $\mathbf{D}_n$ has a rank of three (four is not permissible). What we will show now is that when the rotation is around the optical axis, $\mathbf{D}_n$ has a rank of two. To do this, we substitute Θ_i' with (12) into (30) to get

$$\mathbf{D}_n \triangleq \begin{bmatrix} \mathbf{R}_1\Theta_1 & \mathbf{R}_1\Theta_2 & \cdots & \mathbf{R}_1\Theta_n \\ \Theta_1 & \Theta_2 & \cdots & \Theta_n \end{bmatrix}. \tag{31}$$

It is then easy to see that $\mathbf{D}_n$ will have a rank of at most two, irrespective of how many correspondences we have. The correspondences of two space points that are not colinear with the coordinate origin will actually make $\mathbf{D}_n$ have a rank of two. We therefore have the following corollary, which results from Theorem 1 and Huang and Lee's results.

Corollary 1: In the pure rotation case, given $n \geq 3$ noncolinear image point correspondences, if the matrix $\mathbf{D}_n$ has a rank of three, then the rotation is not around the optical axis, and in this case the motion parameters $r_{13}, r_{23}, r_{31}, r_{32}$ can be determined to within a scalar. If the matrix $\mathbf{D}_n$ has a rank of two, then the rotation is around the optical axis and can be uniquely determined. ∎

Although we have a technique to determine if the rotation is around the optical axis, this technique will break down when noise is present in the data. But, if the matrix $\mathbf{D}_n$ under the given noisy data is still ill conditioned for the solution of r_{13}, etc., we can conclude that the rotation is approximately around the optical axis. A solution of more general motion, however, must utilize depth information or multiple-view matching. The next section examines the three-view problem.

IV. The Three-View Problem

In this section, we consider the three-view problem, and we assume pure rotation for simplification.

Ullman [2], Huang and Lee [1], and Aloimonos and Brown [3] all investigated the three-view problem and developed algorithms. A numerical analysis of Ullman's nonlinear algorithm and Huang and Lee's linear algorithm is given in [6]. Our discussion in this section follows Huang and Lee's formulation [1] but will refine their results.

First we must point out that, in any case, if the motion cannot be determined by two-view matching, then there always exist situations where the multiview matching technique will also fail to determine the motion. A physical example is an object that stops moving after the second view or moves back and forth between two positions. Then matching between an infinite number of views does not help to remove the ambiguity of the motion between the first two views.

Now let us consider the three-view problem. The three-view motion problem for pure rotation will generally involve three rotation matrices: $\mathbf{R} = (r_{ij})$, the rotation between the first and the second views, $\mathbf{S} = (s_{ij})$, the rotation between the second and the third views, and $\mathbf{W} = (w_{ij})$, the rotation between the first and the third views. The goal is to determine both $\mathbf{R}$ and $\mathbf{S}$ from three-view matching. Physically we can see that if one of $\mathbf{R}$ and $\mathbf{S}$ describes a rotation around the optical axis, then the other one that is not around the optical axis will not be determined. Otherwise, we can solve the motion problem without using three or more views by simply rotating the image around the optical axis artificially to get a pseudosequence of images and the correspondences. This is of course impossible. So our analysis of the three-view problem must conform to this intuition.

For simplicity, our proof below is based on one particular algorithm,[1] but the conclusion holds for any algorithm. The same conclusion can also be proven with a more complicated method.

Let $\mathbf{M} = \{13, 23, 31, 32\}$ denote the index set of interest. Assume that $r_{ij}, s_{ij}, w_{ij}, ij \in \mathbf{M}$ have been determined to within a scalar:

$$\begin{aligned} r_{ij} &= \alpha f_{ij} \\ s_{ij} &= \beta g_{ij} \\ w_{ij} &= \gamma h_{ij} \\ ij &\in \mathbf{M} \end{aligned} \tag{32}$$

where f_{ij}, g_{ij}, h_{ij} are known but not α, β, γ. The goal is to determine α, β, γ from three-view matching. The information we have is the following equation [1]:

$$\mathbf{W} = \mathbf{SR}. \tag{33}$$

Huang and Lee [1] have shown that when

$$r_{13}s_{32} - r_{23}s_{31} = 0 \tag{34}$$

does not hold, the three-view problem is uniquely determined; they also considered that when (34) holds, the rotation matrices are still uniquely determined. In the following, we will show that, when (34) holds, there exist situations where the rotations are *not* uniquely determined.

Let

$$\mathbf{a}_R = \begin{bmatrix} r_{32} \\ -r_{31} \\ 0 \end{bmatrix}, \quad \mathbf{b}_R = \begin{bmatrix} -r_{23} \\ r_{13} \\ 0 \end{bmatrix},$$
$$\mathbf{a}_S = \begin{bmatrix} s_{32} \\ -s_{31} \\ 0 \end{bmatrix}, \quad \mathbf{b}_S = \begin{bmatrix} -s_{23} \\ s_{13} \\ 0 \end{bmatrix},$$
$$\mathbf{a}_W = \begin{bmatrix} w_{32} \\ -w_{31} \\ 0 \end{bmatrix}, \quad \mathbf{b}_W = \begin{bmatrix} -w_{23} \\ w_{13} \\ 0 \end{bmatrix}. \tag{35}$$

We then have the following identities:

$$\mathbf{Ra}_R = \mathbf{b}_R \tag{36}$$

$$\mathbf{Sa}_S = \mathbf{b}_S \tag{37}$$

$$\mathbf{Wa}_W = \mathbf{b}_W \quad \text{or} \quad \mathbf{SRa}_W = \mathbf{b}_W. \tag{38}$$

Although we can only solve for the vectors in (36)–(38) to within scale factors (see (32)), these equations still hold for the scaled vectors. Thus, in the following, we will assume that the vectors in the above mentioned equations are known and then develop a method to solve for $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{W}$.

Taking the dot products of both sides of (37) and (38) and using the orthonormality of $\mathbf{S}$, we obtain

$$\mathbf{a}_S^T\mathbf{Ra}_W = \mathbf{b}_S \cdot \mathbf{b}_W. \tag{39}$$

Let $\mathbf{U}$ be any known rotation matrix such that $\mathbf{Ub}_R = \mathbf{a}_R$. Then (36) and (39) yield

$$(\mathbf{UR})\mathbf{a}_R = \mathbf{a}_R \tag{40}$$

$$\mathbf{a}_S^T\mathbf{U}^T(\mathbf{UR})\mathbf{a}_W = \mathbf{b}_S \cdot \mathbf{b}_W. \tag{41}$$

Thus, the rotation axis of $\mathbf{UR}$ is $\mathbf{a}_R$, and the rotation angle of $\mathbf{UR}$ can be solved for from (41).

[1] The algorithm in this section is a simplified version of an original proof, inspired by the anonymous reviewers.

To find out the necessary and sufficient condition under which the rotation matrices can be uniquely determined, we need only to find the conditions under which **R** has a unique solution from (40) and (41). We consider three different situations.

1) $a_R = a_S = 0$*:* In this case, **R**, **S**, and **W** are all around the optical axis, and, hence, are uniquely determined.

2) One of $a_R = 0$ *and* $a_S = 0$ *Holds:* In this situation, either **R** or **S** is a rotation around the optical axis, but not both.

First let us note the fact that if one of $\mathbf{a}_W // \mathbf{a}_R$, $\mathbf{a}_S // \mathbf{b}_R$, and $\mathbf{b}_S // \mathbf{b}_W$ holds, so do the other two (see [1, eqs. (30) and (31)]. The parallelness of these three pairs of vectors is equivalent to (34).

Now we show that the rotation around the optical axis is determined, but not the other. We assume, without loss of generality, that $\mathbf{a}_R = 0$. Then **R** is known. Equations (37) and (38) give

$$\mathbf{S}\mathbf{a}_S = \mathbf{b}_S$$
$$\mathbf{S}\mathbf{R}\mathbf{a}_W = \mathbf{b}_W. \tag{42}$$

Then if and only if $\mathbf{a}_S$ and $\mathbf{R}\mathbf{a}_W$ are not parallel to each other, **S** can be uniquely determined from (42). However, when $\mathbf{a}_R = 0$, $\mathbf{b}_S$ is parallel to $\mathbf{b}_W$ in the sense that

$$\mathbf{b}_S \times \mathbf{b}_W = 0. \tag{43}$$

Therefore, in this case, **S** cannot be uniquely determined from (42).

The situation where $\mathbf{a}_S = 0$ is quite similar. Therefore, the rotation around an axis other than the optical axis and hence **W** are undetermined.

3) $a_R \neq 0$ *and* $a_S \neq 0$*:* In this case, in general **R** can be solved for from (40) and (41), and then **S** can be solved for from (37) and (38). We now consider the singular situations where either **R** or **S** cannot be determined.

If $\mathbf{a}_W$ is parallel to $\mathbf{a}_R$, then the rotation angle of **R** cannot be determined from (41). In this case, assume $\mathbf{a}_W = k\mathbf{a}_R$, where k is a constant. Then (37) and (38) give

$$\mathbf{S}\mathbf{a}_S = \mathbf{b}_S$$
$$k\mathbf{S}\mathbf{R}\mathbf{a}_R = k\mathbf{S}\mathbf{b}_R = \mathbf{b}_W. \tag{44}$$

S cannot be uniquely determined from (44) because when $\mathbf{a}_W$ is parallel to $\mathbf{a}_R$, $\mathbf{a}_S$ is also parallel to $\mathbf{b}_R$. Therefore, when $\mathbf{a}_W$ is parallel to $\mathbf{a}_R$, the rotations are undetermined.

Similarly, if $\mathbf{a}_S$ is parallel to $\mathbf{R}\mathbf{a}_W$, then **S** cannot be solved for from (37) and (38). In this case, from (37) and (38) we must also have $\mathbf{b}_S$ in parallel to $\mathbf{b}_W$. This condition is exactly the same as $\mathbf{a}_W // \mathbf{a}_R$, which again is equivalent to (34).

In summary, we conclude that when $\mathbf{a}_R \neq 0$ and $\mathbf{a}_S \neq 0$, then if and only if (34) does not hold, the three-view problem is uniquely determined.

We thus have the following theorem.

Theorem 2: In the three-view problem, if both **R** and **S** are around the optical axis, then the motions are determined. If one of **R** and **S** is around the optical axis and the other is not, then the rotation not around the optical axis is not determined. If both **R** and **S** are not around the optical axis, then if and only if (34) does not hold are the rotations uniquely determined. ∎

V. Summary

In this short paper, we have shown that, under orthographic projection, the two-view motion problem is determined if and only if the rotation is around the optical axis, and the three-view motion problem is determined if and only if some necessary and sufficient conditions are satisfied. These results have updated our understanding about the motion estimation problem under orthographic projection.

Appendix

In this appendix, we derive (26) and (27). Although these two equations can be deduced from the identity $\mathbf{X}'_i \times \mathbf{X}'_j = \mathbf{R}(\mathbf{X}_i \times \mathbf{X}_j)$, where $\mathbf{X}'_i = \mathbf{R}\mathbf{X}_i$ and $\mathbf{X}'_j = \mathbf{R}\mathbf{X}_j$, we will give a simpler derivation here.

We first derive (27). Set $t_1 = t_2 = 0$ in (4) and rewrite (4) as

$$\begin{bmatrix} r_{13} \\ r_{23} \end{bmatrix} Z_i = \begin{bmatrix} x'_i \\ y'_i \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{A1}$$

and

$$\begin{bmatrix} r_{23} \\ -r_{13} \end{bmatrix} Z_j = \begin{bmatrix} y'_j \\ -x'_j \end{bmatrix} + \begin{bmatrix} -r_{21} & -r_{22} \\ r_{11} & r_{12} \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix} \tag{A2}$$

for two different indices i and j. Then taking the dot products of both sides of (A1) and (A2), we have

$$0 = \left(\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \right)^T \cdot \left(\begin{bmatrix} y'_j \\ -x'_j \end{bmatrix} + \begin{bmatrix} -r_{21} & -r_{22} \\ r_{11} & r_{12} \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix} \right). \tag{A3}$$

After some simplification we can get (27). Equation (26) results from a consideration of symmetry. That is, if we start with the motion equation $\mathbf{X} = \mathbf{R}^T\mathbf{X}' - \mathbf{R}^T\mathbf{T}$ and do the same as was done above, we can obtain (26).

It is straightforward that (26) and (27) are redundant equations because the motion equation (2) and the rigidity property of **R** (i.e., the orthonormality and unit determinant property of **R**) are all what we know about rigid motion. However, one can also directly prove that when (16) is satisfied for the correspondences, then (26) and (27) will become trivial equations.

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Detection of Deadlocks in Flexible Manufacturing Cells

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Abstract—**A problem emerging from CIM implementation is that of system deadlock. This short paper introduces deadlocking of manufacturing systems and describes some related work in the deadlocking of computer systems. A formal model for manufacturing systems deadlock detection is presented. Necessary and sufficient conditions for manufacturing system deadlock based on actual manufacturing system characteristics are defined along with a set of bounds for searching for deadlocks. An implementation approach is also presented.**

Keywords—**Deadlock detection, deadlocking, manufacturing control.**

I. Introduction

The scheduling and control of flexible manufacturing systems (FMS's) has received significant attention because of the potential gains that can be had from significant improvements in these areas. Heuristic solutions are often the norm for real-time implementation of shop-floor control activities. Often heuristic solutions lack considerations of overall system implications, and several practical problems that arise in the control aspect of unmanned FMS's have not been studied. In this short paper, we address one specific problem of control, namely, system deadlock that can arise in an unmanned FMS. The intent of this work is to describe the system deadlock problem as it applies to FMS control and establish its credibility as a problem area of both theoretical and practical interest. The problem of FMS deadlock has been ignored by most research in scheduling and control.

An FMS is in a state of deadlock when parts are assigned to various machines in a manufacturing system such that any further part flow is inhibited. Deadlocks can occur in any "direct-address" FMS. A "direct-address" FMS is one that employs a direct-address material-handling device such as a robot or a shuttle cart (as opposed to a material-handling system like a recirculating conveyor). This configuration is often used in an FMS where a robotic device is used to service several machines in an unmanned setting (Fig. 1). Fig. 1 shows one possible configuration of a "direct-address" FMS. A single robot is used to load/unload parts and to move parts for processing between the various machines in the system. There is no buffer or auxiliary storage device in the system. The control of the unmanned cell is executed by a control computer whose function is to coordinate and plan movement of parts in the system.

Control in an unmanned FMS is usually implemented using state tables, where the actions to be taken are implemented based on the "state" of the cell and the incoming requests from the various machines in the cell. In such a control system, suppose the following situation arises: part 1 is loaded at machine A, part 2 is loaded at machine B, and the robot is idle. On completion of processing part 1 at machine A, a command is sent to the control computer indicating completion of machining. The control system could then activate the robot to move part 1 to the next destination determined by the process routing for part 1. If the next destination of part 1 is machine B, and the next destination for part 2 (currently at machine B) is machine A, then the system will be in a state of deadlock.

A solution to resolving the deadlock would be to allow a storage space that could be used to move parts temporarily to alleviate the deadlock. In the above example, part 1 could be moved to the storage space, part 2 moved from B to A, and then part 1 moved from storage to B. However, the presence of a storage space by itself is not sufficient to prevent a deadlock.

Consider a situation with three machines and three parts. Part 1 is currently at machine A, part 2 at machine B, part 3 at machine C, and the next requested machines are C by part 1, C by part 2, and B by part 3. If part 1 finishes processing at machine A, and is moved to the storage space while machine C is still busy, this will lead to a deadlock since no more part movement will be possible. Improper use of the available storage to alleviate deadlocking can also result in a system deadlock.

These examples indicate the relative ease with which deadlocks can occur in an unmanned FMS. The total number of deadlock possibilities in a manufacturing system with n machines is given by $\sum_{i=2}^{n}\binom{n}{i}$, and more than one deadlock can occur simultaneously. The existing approach to the deadlock problem is to consider it during the design phase of an FMS and try to design deadlock-free systems right from the beginning. Two approaches often used to design deadlock free systems are:

1) ensuring that all parts flow in the same direction (this limits the type of parts that can be processed) and
2) batching the parts waiting to be processed according to their flow direction [2]. Once the parts are batched, the manufacturing system can then process one unidirectional batch at a time. This reduces total machine utilization.

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