

# Elliptical Gaussian Filters

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## Abstract

*A multiscale region detector for low-level image analysis is described. The basis of the detector is a set of filters similar to the Laplacian of an elliptical Gaussian. The responses of these filters to ideal ellipses are derived, and equations for determining the parameters of detected ellipses from the filter responses are found. The use of scale-space techniques to eliminate false ellipse sites in real images is described.*

## 1. Introduction

Many image analysis tasks require regions of relatively uniform gray level to be extracted from an image. One such task is texture analysis, in which regions may have arbitrary density, size, and contrast which are *a priori* unknown [1, 2]. Blostein and Ahuja [3, 4] approach this problem by approximating texture elements, or *texels*, as collections of overlapping disks. Their detector searches the image for circular regions of relatively uniform intensity. They use multiple filters to find disks at a range of sizes, combining overlapping disks of similar contrast values to form the final regions. They demonstrate the effectiveness of this approach over a wide range of images, but they acknowledge that their algorithm is not well-suited to finding long, thin regions and that "the use of primitives with elongated shapes could lead to better detection of elongated regions."

The region detector we propose is designed to improve upon the results of Blostein and Ahuja. Rather than using circular disks as primitives, we use ellipses of relatively

uniform gray level. Scale-space concepts are used to fit ellipses over a range of sizes, eccentricities, and orientations. Though discussion of an implementation of this system is beyond the scope of this paper (see [5]), the extracted ellipses may be used by medium-level vision processes for a variety of purposes. For example, arbitrary (non-elliptical) regions can be constructed by combining individual overlapping ellipses with similar contrast values; such a system could be used as a region detector for a variety of applications.

## 2. The Filter Selection Process

Our approach to the problem of region extraction is to find a filter or a set of filters which, when convolved with an image, will identify elliptical regions of nearly constant gray level. Combining ideas from the literature with our own analysis and experimentation leads us to choose filters similar to the Laplacian of an elliptical Gaussian.

### 2.1. Properties of an Ideal Filter

Accurate evaluation of ellipse-detecting filters requires consideration of the ideal properties of such a filter. The properties below are generalized to apply to any uniform *primitive*-detecting filter and are listed in order of decreasing importance.

1. When convolved with an image, the filter should produce a local maximum or minimum at the center of each *primitive*-shaped region of relatively constant

gray level in the image. The convolution of a filter  $H(x, y)$  with an image  $I(x, y)$  is defined by

$$Z(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) I(x - u, y - v) du dv \quad (1)$$

In this notation,  $Z(x, y)$  should have a local maximum or minimum at the center of a *primitive*-shaped area of nearly uniform intensity.

2. The convolution integral given above should have a closed-form solution at the center of a single *primitive*-shaped region of uniform contrast. For a closed region  $R$  of shape *primitive*, the following integral should be analytically solvable:

$$Z = \iint_R H(u, v) I(u, v) du dv \quad (2)$$

3. The function defining the filter and its derivatives should be smooth to minimize the effects of noise.
4. The filter should exhibit *spatial domain localization*. Pixels in the image which are near to the center of convolution should be weighted more heavily than pixels which are far away, ensuring that the filter response at a given location in the image is minimally affected by information in remote areas of the image.
5. The filter should also exhibit *frequency domain localization*. Our detection method is based on Witkin's scale-space technique [6], in that we use multiple filters of different shapes, sizes, and orientations to detect elliptical regions in the image. The success of this method is predicated upon having filters which are highly selective of features near their own scales. Consequently, we desire a filter which has a high sensitivity to a narrow band of frequencies and a very low sensitivity to frequencies outside that band.

We cannot expect our filter to fully satisfy all of these requirements. In particular, the demands of spatial and frequency domain localizations conflict, as documented in [7].

## 2.2. Literature Review

In [7], Marr and Hildreth show that the circularly symmetric Gaussian function  $G(r) = \exp(-r^2/2\sigma^2)$ , where  $\sigma$  is the "size" of the filter, optimizes the constraints of spatial and frequency localization described above. It is smooth and continuous, as are all of its derivatives in both the spatial and frequency domains. In addition, there is biological evidence [8] indicating that retinal ganglion cells are

circularly symmetric and that these receptors can be modeled with a difference of two Gaussians, which is a discrete approximation<sup>1</sup> to  $\nabla^2 G$ .

Marr suggests that a set of  $\nabla^2 G$  filters, representing a range of  $\sigma$ , should be applied to an image to detect edges. Witkin [6] develops this idea further by introducing the concept of a *scale-space*. He proposes that  $\nabla^2 G$  filters of a continuous range of sizes be applied to an image. The zero-crossings of the responses for one-dimensional signals are recorded in a two-dimensional  $x$ - $\sigma$  space. In this way, the locations of edges in an image can be accurately located by tracing zero-crossings at a large filter size through applications of successively smaller filters.

The scale-space concept is extended to the problem of region detection by Crowley [9]. Whereas Witkin analyzes the zero-crossings of  $\nabla^2 G$  over a range of scales, Crowley examines the peaks in the responses. A peak in the  $\nabla^2 G$  response signifies the center of a circular region of relatively constant intensity. The size of this region depends on the size of the filter producing the peak. Thus disks of a range of sizes are found by applying a range of filters to an image.

Blostein and Ahuja apply these principles to develop their region extractor, which uses a circular disk as the shape primitive. They derive the response of a  $\nabla^2 G$  filter to an ideal disk of diameter  $D$  and contrast  $C$ . They also find the response due to a second filter,  $\frac{\partial}{\partial \sigma} \nabla^2 G$ . Application of this filter implicitly incorporates Witkin's scale-space technique by using information across scales (varying  $\sigma$ ). The  $\nabla^2 G$  and the  $\frac{\partial}{\partial \sigma} \nabla^2 G$  filter responses are two equations in two unknowns ( $C$  and  $D$ ) and can be solved for these parameters.

Combining these ideas has led us to seek a set of filters, each of which is tuned to produce maximal response for an ellipse of a particular size, shape, and orientation.

## 2.3. From Circles to Ellipses

Since an ellipse is in some sense a generalized version of a circle, our approach is to generalize Blostein and Ahuja's  $\nabla^2 G$  filter to respond maximally to ellipses rather than circles. An elliptical Gaussian is similar to the circularly symmetric version except that its level curves are ellipses. The  $y = 0$  plane contains a Gaussian curve of size  $\sigma_x$  while the  $x = 0$  plane contains a Gaussian curve of size  $\sigma_y$ . This function is defined by

$$F(x, y) = \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) \quad (3)$$

and the Laplacian of this function is given by

$$\nabla^2 F = \left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} - \frac{x^2}{\sigma_x^4} - \frac{y^2}{\sigma_y^4}\right) e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} \quad (4)$$

<sup>1</sup>The difference of two Gaussians is a discrete approximation to  $\frac{\partial}{\partial \sigma} G$ , which is in turn related to  $\nabla^2 G$  by the diffusion equation  $\nabla^2 G = \frac{1}{\sigma} \frac{\partial}{\partial \sigma} G$ .

We define the image  $I(x, y)$  to contain only a single ellipse of contrast  $C$  and major and minor axes  $A$  and  $B$ , respectively:

$$I(x, y) = \begin{cases} C & \text{if } \frac{x^2}{A^2} + \frac{y^2}{B^2} \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Requirement 2 of Section 2.1 is that the convolution  $\nabla^2 F * I$  must have a closed-form solution at the center of the ellipse. Unfortunately, we have not succeeded in finding a closed-form solution for this convolution.

In fact, on further inspection, we can see that although  $F(x, y)$  is elliptically symmetric (that is, its level curves are ellipses),  $\nabla^2 F$  is not. Furthermore, an elliptically symmetric version of  $\nabla^2 F$  will permit a closed-form solution of the convolution integral, since a variable substitution can be made which maps the ellipses into disks. This analysis has led us to define the following filter, which is "nearly" the Laplacian of an elliptical Gaussian:

$$\mathcal{E}(x, y) = \frac{1}{\sigma_x \sigma_y} \left( 1 - \frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right) e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} \quad (6)$$

We show in Section 3 that by varying  $\sigma_x$  and  $\sigma_y$  in  $\mathcal{E}(x, y)$  we obtain a set of filters which respond maximally to ellipses of a range of sizes and eccentricities. We also find a closed-form solution for the response of  $\mathcal{E}(x, y)$  to an ideal elliptical region of constant gray level. Thus, the essential two requirements for our filter are satisfied. By inspection, the filter function is smooth, and so are its derivatives. It exponentially decreases towards zero in all directions in both the spatial and the frequency domains, thereby satisfying the desired localization property.

### 3. Derivation of Filter Responses

In this section we build the mathematical foundation upon which our region extraction algorithm is based. We derive the responses of several filters to an image containing only a single isolated ellipse of constant intensity. While this ideal case is not likely to be encountered in any real images, the results obtained here must be used as a reference when one attempts to locate nonideal ellipses in an image.

#### 3.1. The Elliptical Gaussian Filter

We have already claimed that the filter  $\mathcal{E}(x, y)$  is suitable for detecting elliptically shaped regions of relatively constant gray level. Here, we derive the response of this filter to an ideal ellipse of uniform contrast. In the next section, we will use this response to verify that this filter is suitable for ellipse detection.

We defined our ideal ellipse in Equation (5). The convolution of the filter  $\mathcal{E}(x, y)$  with this image  $I(x, y)$  is

$$Z(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma_x \sigma_y} \left( 1 - \frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) \exp \left( -\frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) I(x-u, y-v) du dv \quad (7)$$

Our efforts to find a closed form for this integral have been unsuccessful. However, we can still obtain a useful result by making two assumptions which are reasonable when considered in the context of an actual implementation. We assume:

1. The filter is being applied at the center of an ellipse, i.e., at  $(x, y) = (0, 0)$ . This is justified because the filter response is only extremal at ellipse centers and because in our implementation we attempt to fit ellipses only at local extrema of the convolution result.
2. The filter has the same eccentricity and orientation as the ellipse, though its scale relative to the ellipse may vary. Again, since the filter only responds maximally to ellipses with these properties and since maximal response is required for an ellipse to be detected, this assumption is acceptable.

The first assumption enforces a relationship between the location of the image ellipse and the location of the filter: the origins of both must coincide. This allows us to set  $(x, y) = (0, 0)$  in our convolution equation. The result is a double integral which simply finds the volume under the portion of the  $\mathcal{E}(x, y)$  surface bounded by the ideal ellipse (multiplied by the scaling factor  $C$ ):

$$Z = C \int_{-A}^A \int_{-B\sqrt{1-\frac{u^2}{A^2}}}^{B\sqrt{1-\frac{u^2}{A^2}}} \frac{1}{\sigma_x \sigma_y} \left( 1 - \frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) \exp \left( -\frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) dv du \quad (8)$$

The second assumption enforces a relationship between the eccentricity of the image ellipse and the eccentricity of the level curves of the filter: they must be similar. Mathematically, this means that we can use a scaling factor  $s$  which is defined by  $s = A/\sigma_x = B/\sigma_y$ . Setting  $A = s\sigma_x$  and  $B = s\sigma_y$  in our integral yields

$$Z = C \int_{-s\sigma_x}^{s\sigma_x} \int_{-s\sigma_y\sqrt{1-\frac{u^2}{s^2\sigma_x^2}}}^{s\sigma_y\sqrt{1-\frac{u^2}{s^2\sigma_x^2}}} \frac{1}{\sigma_x \sigma_y} \left( 1 - \frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) \exp \left( -\frac{u^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_y^2} \right) dv du \quad (9)$$

Now we have a form which can be solved with the help of two variable substitutions. We first map the ellipses into

circles by scaling the axes. We let  $x = u/\sigma_x$ ,  $y = v/\sigma_y$  and find that the Jacobean gives us  $du dv = \sigma_x \sigma_y dx dy$ . Making these substitutions, we have

$$Z = C \int_{-s}^s \int_{-s}^s \sqrt{1 - \frac{x^2}{s^2}} \left(1 - \frac{x^2}{2} - \frac{y^2}{2}\right) \exp\left(-\frac{x^2}{2} - \frac{y^2}{2}\right) dy dx \quad (10)$$

We map into polar coordinates by letting  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and  $dx dy = r dr d\theta$ . Performing the substitution and adjusting the limits accordingly, we obtain

$$Z = C \int_{-\pi}^{\pi} \int_{-s}^s \left(1 - \frac{r^2}{2}\right) e^{-r^2/2} r dr d\theta \quad (11)$$

Integrating by parts on each piece of the integral and combining terms yield the following solution:

$$Z = \pi C s^2 e^{-s^2/2} \quad (12)$$

### 3.2. Verifying That This Filter Detects Ellipses

Having derived an equation for the response of  $\mathcal{E}(x, y)$  to an ideal ellipse, we now verify that this filter is acceptable as an ellipse detector.

We have asserted that  $\mathcal{E}(x, y)$  produces a maximal (minimal) response when applied to the center of an elliptically shaped region of uniform positive (negative) contrast. To verify this mathematically, we use the following approach. We have defined the filter to be a function of  $\sigma_x$  and  $\sigma_y$ . We can change the filter size by changing these parameters. For a given ellipse of major axis  $\hat{A}$  and minor axis  $\hat{B}$  (oriented in standard position) there is at least one filter (we will show there is only one) which produces the maximum response to that ellipse. Denote the parameters of that filter by  $\hat{\sigma}_x$  and  $\hat{\sigma}_y$ . We must then verify the following consistency property:

If the maximum response to an ellipse of size  $(\hat{A}, \hat{B})$  is found with a filter of size  $(\hat{\sigma}_x, \hat{\sigma}_y)$ , then a filter of size  $(\hat{\sigma}_x, \hat{\sigma}_y)$  gives the maximum response when applied to an ellipse of size  $(\hat{A}, \hat{B})$ .

We can check that this consistency holds for  $\mathcal{E}(x, y)$  using Equation (12). By differentiating  $Z$  with respect to each of the variables  $A$  and  $\sigma_x$  we obtain:

$$\frac{\partial Z}{\partial A} = \frac{\partial Z}{\partial s} \frac{\partial s}{\partial A} = \left(\pi C (2s - s^3) e^{-s^2/2}\right) \left(\frac{1}{\sigma_x}\right) \quad (13)$$

and

$$\frac{\partial Z}{\partial \sigma_x} = \frac{\partial Z}{\partial s} \frac{\partial s}{\partial \sigma_x} = \left(\pi C (2s - s^3) e^{-s^2/2}\right) \left(\frac{-A}{\sigma_x^2}\right) \quad (14)$$

Setting both of these to zero to find local extrema yields, by inspection,

$$\frac{\partial Z}{\partial A} = 0 \iff s = \sqrt{2} \quad (15)$$

$$\frac{\partial Z}{\partial \sigma_x} = 0 \iff s = \sqrt{2} \quad (16)$$

which is the desired consistent relationship. In a similar way, it can be shown that  $\frac{\partial Z}{\partial B}$  and  $\frac{\partial Z}{\partial \sigma_y}$  are zero when  $s = \sqrt{2}$  as well.

We have verified mathematically that for every elliptical region  $\hat{R}$  there is a filter  $\hat{\mathcal{E}}(x, y)$  which responds maximally to it, and, furthermore, that  $\hat{R}$  is the one and only elliptical region which forces a maximal response from this filter. We conclude that passing a single filter  $\mathcal{E}$  of fixed  $\sigma_x$  and  $\sigma_y$  allows us to detect elliptical regions of uniform contrast whose parameters are given by  $(A, B) = (\sigma_x \sqrt{2}, \sigma_y \sqrt{2})$ . We further conclude that by passing a series of such filters, each with different  $\sigma_x$  and  $\sigma_y$ , we extract ellipses of a variety of shapes and sizes.

### 3.3. Recovering Ellipse Parameters

The filter response given in Equation (12) is a function of the contrast  $C$  of the ellipse and the scale factor  $s$  of the ellipse relative to the filter. When the filter is applied to a real image and a response  $Z$  is obtained at a point in the image, it is impossible to determine the parameters of the ellipse from the  $\mathcal{E}(x, y)$  response alone. For example, a given response could be due to a large ellipse of very low contrast or to a small ellipse of very high contrast. In essence, we have one equation  $Z$  with two unknowns  $C$  and  $s$ . We need more information to be able to extract the ellipse parameters from the filter response.

One solution to this problem is to use Witkin's scale-space concepts and gather information about the ellipse from the behavior of the filter response across scales. In Blostein and Ahuja's detector, for example, they implicitly use scale-space information by convolving the image with two filters,  $\nabla^2 G$  and  $\frac{\partial}{\partial \sigma} \nabla^2 G$ . The two filter responses obtained allow them to solve for the two unknowns, the diameter and the contrast of the disk.

We use a similar approach here, except that in addition to the ellipse-detecting filter already discussed we use two additional filters which are related to the original by differentiation with respect to the scale variable. For ease of notation in the following equations, we define a variable  $z$  such that

$$z(x, y) = \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} \quad (17)$$

In terms of  $z$ , these additional filters are defined by

$$\mathcal{E}_1(z) = \frac{2}{\sigma_x \sigma_y} (1 - 3z + z^2) e^{-z} \quad (18)$$

and

$$\mathcal{E}_2(z) = \frac{2}{\sigma_x \sigma_y} (1 - 11z + 11z^2 - 2z^3) e^{-z} \quad (19)$$

These can be shown to be related to the first and second partial derivatives with respect to  $s$  in the following way:

$$\mathcal{E}_1 = s \frac{\partial \mathcal{E}}{\partial s} \quad (20)$$

and

$$\mathcal{E}_2 = s^2 \frac{\partial^2 \mathcal{E}}{\partial s^2} \quad (21)$$

By linearity of the differentiation operation, the filter responses of each of these filters can be quickly obtained from the response  $Z$  to  $\mathcal{E}(x, y)$ . The response to  $\mathcal{E}_1(x, y)$ , denoted by  $Z_1$ , is

$$Z_1 = s \frac{\partial Z}{\partial s} = \pi C s^2 (2 - s^2) e^{-s^2/2} \quad (22)$$

The response to  $\mathcal{E}_2(x, y)$ , denoted by  $Z_2$ , is

$$Z_2 = s^2 \frac{\partial^2 Z}{\partial s^2} = \pi C s^2 (2 - 5s^2 + s^4) e^{-s^2/2} \quad (23)$$

### 3.4. Calculations of $C$ and $s$ from $\mathcal{E}$ , $\mathcal{E}_1$ and $\mathcal{E}_2$

We have three equations (the responses of three filters) in two unknowns,  $C$  and  $s$ . While this is an overdetermined system, we have found through implementation and analysis that three filters provide additional information about the candidate region which is useful in determining whether the region is sufficiently elliptical. To this end, we derive two values of  $s$  which correspond to two sets of filter responses.

Using  $Z$  from Equation (12) and  $Z_1$  from Equation (22), we can eliminate  $C$  by division and solve for  $s$  immediately:

$$s_1 = \sqrt{2 - \frac{Z_1}{Z}} \quad (24)$$

In similar fashion, using  $Z_2$  from Equation (23) and  $Z$ , we can solve for another version of  $s$  which we call  $s_2$ :

$$s_2 = \sqrt{\frac{5}{2} - \frac{1}{2} \sqrt{17 + \frac{4Z_2}{Z}}} \quad (25)$$

These two values  $s_1$  and  $s_2$  are used in an implementation to eliminate false ellipse sites (local maxima of the filter response) if they differ by a large factor. If they are sufficiently close in value, they can be combined (by either simple or weighted averaging) to find a single value  $s$  which will be considered the actual scale factor of the ellipse relative to the filter. The major and minor axis dimensions of the ellipse are then given by:

$$A = s\sigma_x \quad (26)$$

$$B = s\sigma_y \quad (27)$$

Finally, by substituting this  $s$  into the formula for  $Z$  and solving, the ellipse contrast  $C$  can be determined:

$$C = \frac{Z}{\pi s^2} e^{s^2/2} \quad (28)$$

### 3.5. Verification of the Derived Equations

To verify the accuracy of these derivations and the suitability of this method for image analysis, an implementation of this region detection system has been applied to synthetic and real images. The synthetic image experiments show that the region detector is robust with respect to image noise and shape deviations. The natural image experiments show the strength of the detector in modeling elongated homogeneous regions. These experiments and results are described in [5].

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