Necessary and Sufficient Conditions for a Unique Solution of Plane Motion and Structure

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ABSTRACT. First, we show an uncertain situation for the two-view plane motion problem in which an infinite number of solutions may result. Then, we present necessary and sufficient conditions for a unique solution of plane motion and structure from any number of views. These algorithm-independent conditions enhance our understanding about the problem of estimating plane motion and structure from image sequences and may be incorporated in any practical algorithms for estimating the motion and structure of a planar surface.

1 Introduction

Although many plane motion algorithms have been developed (e.g., [9], [11], [6], [3], [7], and [1]), the conditions obtained for a unique solution of motion and structure of a planar surface ([9], [7], [11], [1], [3]) are restricted to only two and three views. Even for two and three views, the existing conditions are not complete in the sense that some rare situations have been ignored, as will be discussed here.

This paper presents necessary and sufficient conditions for a unique solution of the motion and structure of a planar surface from any number of views under perspective projection. First, we show an uncertain situation in the two-view problem in which an infinite number of solutions result. Though this uncertain situation rarely occurs for opaque planes, it has to be considered when some of the images are observed through a mirror. The knowledge of this uncertain situation therefore help analyze mirror-reflected images. Then, we present necessary and sufficient conditions for a unique solution of the motion and structure from multiple images. These conditions apply to any number of images and are algorithm-independent.

2 The Uncertain Situation in the Two-View Problem

A plane can be described by an equation as follows:

$$\mathbf{N}^T\mathbf{X} = [n_1 \quad n_2 \quad n_3]\mathbf{X} = 1, \tag{2-1}$$

where $\mathbf{X} = [X, Y, Z]^T$ is a point on the plane, and $\mathbf{N}$ is the plane parameters which are used to represent the plane. A *degenerate* case occurs when a plane has an equation of $\mathbf{N}^T\mathbf{X} = 0$. In this case, the projection of the plane in the image plane is a line. Degenerate views can be removed from the image sequence, since it is difficult to obtain point correspondences when the projection of a plane is a line. Therefore, in

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the following, we assume that the projection of a plane in any view is not degenerate. Using the conventional perspective projection model, the projection of a space point (X,Y,Z) in the image plane is determined by $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$. Let $\Theta = [x, y, 1]^T$; then $\mathbf{X} = \Theta Z$. Similarly, if $\mathbf{X}' = [X', Y', Z']^T$; then $\mathbf{X}' = \Theta' Z'$, where $\Theta' = [x', y', 1]^T$.

Let $\mathbf{X}$ be a point before the motion and $\mathbf{X}'$ be the position of the point after motion, then $\mathbf{X}'$ and $\mathbf{X}$ are related by the following equation

$$\mathbf{X}' = \Theta' Z' = \mathbf{R}\mathbf{X}+\mathbf{T} = \mathbf{R}\mathbf{X}+\mathbf{T}\mathbf{N}^T\mathbf{X} = \mathbf{K}\Theta Z \quad , \tag{2-2}$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{T}$ a translation vector, and

$$\mathbf{K} = \mathbf{R}+\mathbf{T}\mathbf{N}^T. \tag{2-3}$$

Matrix $\mathbf{K}$ of the form (2-3) is called the *plane motion matrix* (PMM), and $\mathbf{R}$, $\mathbf{T}$ and $\mathbf{N}$ are called a *plane motion decomposition* (PMD) of $\mathbf{K}$.

From Equation (2-2), we have the following equation:

$$\Theta' \times \mathbf{K}\Theta = 0. \tag{2-4}$$

Matrix $\mathbf{K}$ can be estimated to within a scalar from Equation (2-4) with four correspondences of points if and only if no three of the points are colinear in the space ([9], [11], [3]). The scalar can be determined by the rigidity condition ([9], [11], [3], [1]). Thus the major problem left is to solve for $\mathbf{R}$, $\mathbf{T}$ and $\mathbf{N}$ from $\mathbf{K}$. We can determine only the direction of $\mathbf{T}$, $\mathbf{N}$, and the combined magnitude $\|\mathbf{N}\|\,\|\mathbf{T}\|$. Therefore, a PMD of $\mathbf{K}$ is distinct if and only if $\mathbf{R}$, or the direction of $\mathbf{T}$ or $\mathbf{N}$, is distinct.

Tsai, Huang, and Zhu established the following results (see also [8], [7], [1], and [11] for different versions) regarding the uniqueness of solution:

1. if the singular values of $\mathbf{K}$ are all distinct, then $\mathbf{K}$ has two PMDs.
2. if two of the singular values of $\mathbf{K}$ are identical, then $\mathbf{K}$ has a unique PMD.
3. if all three singular values of $\mathbf{K}$ are identical, then $\mathbf{K}$ has a unique PMD; in this case, $\mathbf{R} = \mathbf{K}$, $\mathbf{T} = 0$, and $\mathbf{N}$ is undetermined.†

We now show that there exists an uncertain situation in which $\mathbf{K}$ has an infinite number of PMDs and the third conclusion above should be modified as follows:

3*. if all three singular values of $\mathbf{K}$ are identical, then, (a) $\mathbf{K}$ has a unique PMD if $\det(\mathbf{K}) = 1$; in this case, $\mathbf{R} = \mathbf{K}$, $\mathbf{T} = 0$, and $\mathbf{N}$ is undetermined; (b) $\mathbf{K}$ has an infinite number of PMDs if $\det(\mathbf{K}) = -1$.

First, we have the following lemma.

Lemma 2.1. (See also [1].) If $\mathbf{K}$ is a plane motion matrix, the three eigenvalues λ_1, λ_2, and λ_3 of the matrix $\mathbf{K}^T\mathbf{K}$, with $\lambda_3 \le \lambda_2 \le \lambda_1$, satisfy

$$0 \le \lambda_3 \le \lambda_2 = 1 \le \lambda_1 . \tag{2-5}$$

Proof: Since the eigenvalues of $\mathbf{K}^T\mathbf{K}$ must be nonnegative, we need only show that $\lambda_2 = 1$. First we show that 1 is an eigenvalue of $\mathbf{K}^T\mathbf{K}$. Assume $\mathbf{R}$, $\mathbf{T}$ and $\mathbf{N}$ constitute one PMD of $\mathbf{K}$. For any nonzero vector $\mathbf{X}$ that satisfies

$$\mathbf{N}^T\mathbf{X} = 0, \quad \text{and} \quad \mathbf{T}^T\mathbf{R}\mathbf{X} = 0, \tag{2-6}$$

using Equations (2-6) and the orthonormality of $\mathbf{R}$, we have

$$\mathbf{K}^T\mathbf{K}\mathbf{X} = (\mathbf{R}^T + \mathbf{N}\mathbf{T}^T)(\mathbf{R} + \mathbf{T}\mathbf{N}^T)\mathbf{X} = (\mathbf{R}^T + \mathbf{N}\mathbf{T}^T)\mathbf{R}\mathbf{X} = \mathbf{X} = 1 \cdot \mathbf{X}, \quad (2\text{-}7)$$

proving that $\lambda = 1$ is an eigenvalue. Let it be λ_2. Then we show that one of the two remaining eigenvalues, e.g., λ_3 , must be ≤ 1, and the other, λ_1, must be ≥ 1. Let $\mathbf{W}$ be an orthonormal matrix such that

$$\mathbf{W}^T\mathbf{K}^T\mathbf{K}\mathbf{W} = diag(\,\lambda_1 \quad \lambda_2 \quad \lambda_3\,), \quad (2\text{-}8)$$

and let $\mathbf{X}$ be any nonzero vector orthogonal to $\mathbf{N}$, i.e.,$\mathbf{N}^T\mathbf{X} = 0$, then $\mathbf{KX} = \mathbf{RX}$. The orthonormality of $\mathbf{W}$ and $\mathbf{R}$ gives

$$\mathbf{W}^T\mathbf{W} = \mathbf{R}^T\mathbf{R} = \mathbf{I}\,. \quad (2\text{-}9)$$

Now let $\mathbf{U} = [\, u_1 \; u_2 \; u_3 \,]^T = \mathbf{W}^T\mathbf{X}$, and hence $\mathbf{X} = \mathbf{WU}$. Then,

$$\mathbf{X}^T\mathbf{K}^T\mathbf{K}\mathbf{X} = \mathbf{X}^T\mathbf{R}^T\mathbf{R}\mathbf{X} = \mathbf{X}^T\mathbf{X}, \text{ or} \quad (2\text{-}10)$$

$$\mathbf{U}^T\mathbf{W}^T\mathbf{K}^T\mathbf{K}\mathbf{W}\mathbf{U} = \mathbf{U}^T\mathbf{W}^T\mathbf{W}\mathbf{U}. \quad (2\text{-}11)$$

From Equations (2-8), (2-9), and (2-11) we have

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 = u_1^2 + u_2^2 + u_3^2, \text{ or} \quad (2\text{-}12)$$

$$(1 - \lambda_3)u_3^2 = (\lambda_1 - 1)u_1^2\,. \quad (2\text{-}13)$$

Because $\lambda_1 \geq \lambda_3 \geq 0$, to make (2-13) hold, we must have $\lambda_1 \geq 1$ and $\lambda_3 \leq 1$. Q.E.D.

We immediately have the following corollary, due to the fact that the singular values of $\mathbf{K}$ are the nonnegative square roots of the eigenvalues of $\mathbf{K}^T\mathbf{K}$.

Corollary 2.1. One of the singular values of $\mathbf{K}$ must be 1. †

The uncertain situation occurs when all three singular values of $\mathbf{K}$ are identical and det($\mathbf{K}$) = −1. In this case, the motion is a mirror reflection. To show that in this case $\mathbf{K}$ has an infinite number of PMDs, we need only note that the singular value decomposition of $\mathbf{K}$ is not unique. If all three singular values of $\mathbf{K}$ are identical, then, according to Corollary 2.1, they must all be 1. Assume that $\mathbf{K}$ has a singular value decomposition (SVD) as follows:

$$\mathbf{K} = \mathbf{U}\mathbf{I}\mathbf{V}^T = \mathbf{U}\mathbf{V}^T, \quad (2\text{-}14)$$

in which $\mathbf{U}$ and $\mathbf{V}$ are two orthonormal matrices and $\mathbf{I}$ is a 3×3 identity matrix. Then for any orthonormal matrix $\mathbf{P}$, we have

$$\mathbf{K} = \mathbf{U}\mathbf{P}\mathbf{P}^T\mathbf{V}^T = (\mathbf{U}\mathbf{P})\mathbf{I}(\mathbf{V}\mathbf{P})^T, \quad (2\text{-}15)$$

in which $\mathbf{UP}$ and $\mathbf{VP}$ are also orthonormal matrices. Therefore, according to the algorithm of Tsai et al. ([9]), which depends on a unique SVD of $\mathbf{K}$, $\mathbf{K}$ has an infinite number of PMDs. A numerical example is $\mathbf{K} = diag\,(-1, 1, 1)$.

A mirror reflection can occur in only two situations: either the camera catches one of the images through a mirror, or the plane has turned around such that the camera catches its back rather than its face in the second image (implying that the object must be transparent). So in practice, unless a mirror is involved, the uncertain situation rarely occurs for opaque object. However, mirror images are frequently seen in daily life. The understanding of this situation therefore help analyze images observed through a mirror.

When $\mathbf{K}$ is not orthonormal, $\mathbf{K}$ always has two sets of PMDs. To remove the ambiguity, we have to resort to other ways such as using positive depth constraint ([8]), small rotation constraint ([3]), multiple planes, or multiple views. The multiview method in the next section will involve decomposing many two-view PMMs. Though there exist other algorithms ([9], [11], [1], [8]) for decomposing a PMM, we shall use the following simpler method ([3]).

Lemma 2.2. If a PMM $\mathbf{K}$ is not orthonormal and $\mathbf{W}$ is an orthonormal matrix that transforms $\mathbf{K}^T\mathbf{K}$ into the diagonal form in Equation (2-8), then the plane normal vector can be determined to two solutions, each up to a scalar, as follows:

$$\mathrm{N}_{1,2}^{\mathrm{T}} = [\,\sqrt{\lambda_1 - 1} \quad 0 \quad \pm\sqrt{1-\lambda_3}\,]\mathbf{W}^{\mathrm{T}}\,. \tag{2-16}$$

Proof: Consider any point $\mathbf{X}$ that satisfies $\mathrm{N}^{\mathrm{T}}\mathbf{X} = 0$, where $\mathbf{N}$ is the plane normal to be determined. Let $\mathbf{X} = \mathbf{WU}$ with $\mathbf{U} = [\, u_1 \ u_2 \ u_3\,]^{\mathrm{T}}$; then from the intermediate results of Lemma 2.1, we know that $\mathbf{U}$ must satisfy Equation (2-13), or equivalently,

$$[\,\sqrt{\lambda_1 - 1} \quad 0 \quad \pm\sqrt{1-\lambda_3}\,]\mathbf{U} = 0\,. \tag{2-17}$$

from which we have

$$[\,\sqrt{\lambda_1 - 1} \quad 0 \quad \pm\sqrt{1-\lambda_3}\,]\mathbf{W}^{\mathrm{T}}\mathbf{X} = 0\,. \tag{2-18}$$

Comparing $\mathrm{N}^{\mathrm{T}}\mathbf{X} = 0$, and the above equation, we know that the plane normal can be determined to two solutions, each up to a scalar, from Equation (2-16). Q.E.D.

After a solution of the plane normal $\mathbf{N}$ is obtained, the rotation matrix and the translation vector associated with $\mathbf{N}$ can be determined in closed form ([11], [3], [2]). For example, a simple, closed-form solution can be obtained as follows ([3], [2]):

$$\mathrm{R} = [\,\mathbf{Y}_1 \quad \mathbf{Y}_2 \quad \mathbf{Y}_1\times\mathbf{Y}_2\,][\,\mathbf{X}_1 \quad \mathbf{X}_2 \quad \mathbf{X}_1\times\mathbf{X}_2\,]^{-1}, \quad \mathrm{T} = (\mathrm{K}-\mathrm{R})\mathrm{N}/||\mathrm{N}||^2, \tag{2-19}$$

where $\mathbf{Y}_i = \mathbf{KX}_i$, $i = 1,2$, and $\mathbf{X}_i$, $i = 1,2$ are any two vectors satisfying

$$\mathbf{X}_1 \perp \mathbf{X}_2 \ \text{ and } \ \mathbf{X}_1\times\mathbf{X}_2 = \mathrm{N} \tag{2-20}$$

3 The Multiview Problem

The multiview problem is important, not only because a spurious solution frequently results in the two-view algorithms, but because the accuracy of the estimation can be improved when a long sequence of images are used. Tsai and Huang ([10]) established a uniqueness theorem for the three-view problem. Their proof and conclusion appear to apply to a special case, as indicated by numerical results and strict proofs ([2]). In the following, we shall obtain necessary and sufficient conditions for a unique solution of motion and structure from any number of images, assuming that no additional constraints are used.

First, we have the following lemmata about the plane motion matrix.

Lemma 3.1. A PMM $\mathbf{K}$ is rank reduced if and only if the projection of the plane after motion is a line in the image plane.

Proof: If $\mathbf{K}$ is rank reduced, then, there exists a nonzero vector $\mathrm{N}^{\mathrm{T}\prime}$ such that $\mathrm{N}'^{\mathrm{T}}\mathbf{K} = 0$. Assume that $\mathbf{X}$ is an arbitrary point on the plane defined by Equation (2-1) and $\mathbf{X}'$ is the correspondence of $\mathbf{X}$. Then, $\mathbf{X}' = \mathbf{KX}$. We therefore have

$$\mathbf{N}'^{T}\mathbf{X}' = \mathbf{N}'^{T}\mathbf{K}\mathbf{X} = 0, \text{ or } \mathbf{N}'^{T}\Theta' = \mathbf{N}'^{T}[x' \quad y' \quad 1]^{T} = 0. \tag{3-1}$$

The above equation gives a line equation in the new image plane.

The sufficient part can be proven by contradiction. Assume that the projection of the plane of Equation (2-1) in the new image plane is a line and **K** has a full rank. Then from Equation (2-2) we have

$$\mathbf{X} = \mathbf{K}^{-1}\mathbf{X}' . \tag{3-2}$$

Thus from (2-1) we have the new plane equation after motion as

$$\mathbf{N}^{T}\mathbf{K}^{-1}\mathbf{X}' = 1 , \tag{3-3}$$

whose projection is not a line in the image plane, contradicting the assumption. Therefore **K** must be rank reduced. Q.E.D.

We immediately have the following corollary:

Corollary 3.1. When a plane motion matrix **K** has a full rank, the plane normal **N** after the motion will be $(\mathbf{K}^{T})^{-1}\mathbf{N}$. †

To ensure that all plane motion matrices have full rank, we need only to remove degenerate views in which the projection of the plane is a line. The situation in which some plane motion matrices are rank reduced is complicated. In practice, degenerate views do not help much in determining the motion and structure of the plane since it is difficult to obtain correspondences for degenerate views. Therefore, we will assume in this section that all plane motion matrices have full rank.

Lemma 3.2. If a full-rank plane motion matrix **K** has a PMD of the form

$$\mathbf{K} = \mathbf{R} + \mathbf{T}\mathbf{N}^{T}, \tag{3-4}$$

then

$$\mathbf{K}^{-1} = \mathbf{R}^{-1} - (\mathbf{R}^{-1}\mathbf{T})(\mathbf{N}^{T}\mathbf{K}^{-1}). \tag{3-5}$$

Proof: Using $\mathbf{K}^{-1}$ to postmultiply Equation (3-4) and then using $\mathbf{R}^{-1}$ to premultiply the resulting equation, we obtain

$$\mathbf{R}^{-1} = \mathbf{K}^{-1} + \mathbf{R}^{-1}\mathbf{T}\mathbf{N}^{T}\mathbf{K}^{-1}, \tag{3-6}$$

from which we obtain Equation (3-5). Q.E.D.

The multiview problem generally involves decomposing motion and structure parameters from multiple plane motion matrices. Assume that n (n≥3) views are available. Let $\mathbf{K}_i$ be the plane motion matrix between the *i*th and the first views, i = 2, ..., n. It is clear that if $\mathbf{K}_i$ is an orthonormal matrix, then $\mathbf{K}_i$ is either a mirror reflection matrix or a rotation matrix; in either case, $\mathbf{K}_i$ provides no information about the plane structure and can thus be discarded, without affecting the uniqueness conditions (see [2] for more details). Therefore, in the following, we assume that $\mathbf{K}_i$ is not orthonormal for all i, i = 2, ..., n. (An orthonormal $\mathbf{K}_i$ must be removed.)

From the last section we know that each nonorthonormal $\mathbf{K}_i$ gives up to two solutions of the plane normal in the first view, one of which is the true solution. Clearly, if one of $\mathbf{K}_i$ admits a unique PMD, the multiview problem is uniquely determined. A more general situation occurs when every $\mathbf{K}_i$ admits exactly two PMDs. We now discuss this general situation.

Let $\mathbf{N}_i^{1,2}$ be the two plane normal solutions in the first view from $\mathbf{K}_i$, i = 2, ..., n, respectively. Let the plane normal solutions be indexed as follows. First, label $\mathbf{N}_2^{1,2}$ arbitrarily. That is, let one of the plane normal solutions from $\mathbf{K}_2$ be $\mathbf{L}_2^1$ and the other solution be $\mathbf{L}_2^2$. Now consider the plane normal solutions $\mathbf{N}_3^{1,2}$ from $\mathbf{K}_3$. If

$$min\left(||\mathbf{N}_3^1 \times \mathbf{L}_2^1||^2, ||\mathbf{N}_3^2 \times \mathbf{L}_2^2||^2\right) < min\left(||\mathbf{N}_3^2 \times \mathbf{L}_2^1||^2, ||\mathbf{N}_3^1 \times \mathbf{L}_2^2||^2\right), \quad (3\text{-}7)$$

then let $\mathbf{L}_3^k = \mathbf{N}_3^k$, $k = 1, 2$; otherwise, let $\mathbf{L}_3^1 = \mathbf{N}_3^2$, $\mathbf{L}_3^2 = \mathbf{N}_3^1$. This process continues recursively for other views. That is, for i = 4, ..., n, if

$$min\left(\sum_{j=2}^{i-1} ||\mathbf{N}_i^1 \times \mathbf{L}_j^1||^2, \sum_{j=2}^{i-1} ||\mathbf{N}_i^2 \times \mathbf{L}_j^2||^2\right) < min\left(\sum_{j=2}^{i-1} ||\mathbf{N}_i^2 \times \mathbf{L}_j^1||^2, \sum_{j=2}^{i-1} ||\mathbf{N}_i^1 \times \mathbf{L}_j^2||^2\right), \quad (3\text{-}8)$$

then let $\mathbf{L}_i^k = \mathbf{N}_i^k$, $k = 1, 2$; otherwise, let $\mathbf{L}_i^1 = \mathbf{N}_i^2$, $\mathbf{L}_i^2 = \mathbf{N}_i^1$. Finally, form two matrices using the plane normal vectors as follows:

$$\mathbf{M}_k = [\mathbf{L}_2^k \quad \mathbf{L}_3^k \quad \cdots \quad \mathbf{L}_n^k], \quad k = 1, 2. \quad (3\text{-}9)$$

The next lemma establish a uniqueness condition for the three-view problem.

Lemma 3.3. In the three-view problem, if both $\mathbf{K}_2$ and $\mathbf{K}_3$ are nonorthonormal and $\mathbf{K}_2$ has full-rank, then a unique solution results if and only if the equations in

$$\mathbf{L}_2^1 \times \mathbf{L}_3^1 = 0 \text{ and } \mathbf{L}_2^2 \times \mathbf{L}_3^2 = 0 \quad (3\text{-}10)$$

do not hold simultaneously.

Proof: Since the true plane normal must be a plane normal solution from both $\mathbf{K}_2$ and $\mathbf{K}_3$, one of the equations in (3-10) must hold. Without loss of generality, assume that $\mathbf{L}_2^1$ and $\mathbf{L}_3^1$ correspond to the true solution, and $\mathbf{L}_2^2$ and $\mathbf{L}_3^2$ are spurious solutions. We need prove only that when both equations in (3-10) are satisfied, a motion solution consistent with all three views can be obtained.

Three views give three PMMs $\mathbf{K}_2$, $\mathbf{K}_3$, and $\mathbf{K}_{32}$, which are the PMMs between the first and second views, between the first and third views, and between the second and third views, respectively. Obviously,

$$\mathbf{K}_3 = \mathbf{K}_{32}\mathbf{K}_2, \quad \text{or } \mathbf{K}_{32} = \mathbf{K}_3\mathbf{K}_2^{-1}. \quad (3\text{-}11)$$

Now assume $\mathbf{L}$ ($\mathbf{L} = \mathbf{L}_2^2 = \alpha\mathbf{L}_3^2$ for some constant α according to Equation (3-10)) is a common solution of the plane normal obtained from both $\mathbf{K}_2$ and $\mathbf{K}_3$. Then, according to Corollary 3.1, $\left(\mathbf{K}_2{}^{\mathrm{t}}\right)^{-1}\mathbf{L}$ should be a plane normal solution from $\mathbf{K}_{32}$. Therefore, we need and need only to prove that there exist some rotation matrix $\mathbf{R}_2$ and some vector $\mathbf{T}_2$ such that

$$\mathbf{K}_{32} = \mathbf{R}_2 + \mathbf{T}_2\mathbf{L}^{\mathrm{t}}\mathbf{K}_2^{-1}. \quad (3\text{-}12)$$

When the above equation is satisfied, the plane normal solution $\mathbf{L}$ is consistent with all three views.

Since $\mathbf{L}$ is a solution from both $\mathbf{K}_2$ and $\mathbf{K}_3$, we must have

$$\mathbf{K}_2 = \mathbf{R}_1 + \mathbf{T}_1\mathbf{L}^{\mathrm{t}}, \quad \mathbf{K}_3 = \mathbf{R}_3 + \mathbf{T}_3\mathbf{L}^{\mathrm{t}}, \quad (3\text{-}13)$$

for some rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_3$ and some vectors $\mathbf{T}_1$ and $\mathbf{T}_3$. Using Equation (3-5) we have

$$K_2^{-1} = R_1^{-1} - (R_1^{-1}T_1)(L^T K_2^{-1}). \tag{3-14}$$

Then from Equations (3-11) and (3-12) we have

$$\begin{aligned} K_{32} &= K_3 K_2^{-1} = (R_3 + T_3 L^T) K_2^{-1} = R_3 K_2^{-1} + T_3 L^T K_2^{-1} \\ &= R_3 [R_1^{-1} - (R_1^{-1} T_1)(L^T K_2^{-1})] + T_3 L^T K_2^{-1} \\ &= R_3 R_1^{-1} + (T_3 - R_3 R_1^{-1} T_1)(L^T K_2^{-1}). \end{aligned} \tag{3-15}$$

It is obvious that if we choose

$$R_2 = R_3 R_1^{-1}, \quad T_2 = T_3 - R_3 R_1^{-1} T_1, \tag{3-16}$$

then Equation (3-12) is satisfied. Therefore, if a plane normal solution obtained from $\mathbf{K}_2$ is parallel to a plane normal solution from $\mathbf{K}_3$, a consistent solution of the motion and structure can always be found for the three views. Q.E.D.

The above lemma also states that in the multiview problem, to determine the uniqueness of the solution we need consider only the plane motion matrices related to the first view. Since for each $\mathbf{K}_i$, one plane normal solution admits exactly one motion solution, the overall motion solution for the image sequence is unique if and only if the plane normal solution for the first view is unique (up to a scalar). Therefore, if and only if the plane motion matrices related to the first view admit a unique plane normal solution for the first view, the structure of the plane and the motion over the whole sequence can be determined uniquely.

We now prove the major theorem of this paper.

Theorem 3.1. Assume that each plane motion matrix $\mathbf{K}_i$, i = 2, ..., n, is invertible and has three distinct singular values. Then $\mathbf{K}_i$, i = 2, ..., n, admit a unique solution of the plane structure and motion if and only if exactly one of $\mathbf{M}_1$ and $\mathbf{M}_2$ defined in (3-9) has a rank of 1.

Proof: Since two views determine the plane normal in the first view to two solutions, at most one spurious solution for the motion problem of the whole sequence may result. The true plane normal $\mathbf{N}$ must be one of the solutions from$\mathbf{K}_i$ for every i. Therefore, by the way in which $\mathbf{M}_1$ and $\mathbf{M}_2$ are constructed, one of $\mathbf{M}_1$ and $\mathbf{M}_2$ must contain solely the plane normal solutions corresponding to the true plane normal and the other must contain only the spurious solutions. Without loss of generality, assume $\mathbf{M}_1$ contains the correct solutions and $\mathbf{M}_2$ contains the spurious solutions. It is clear that $\mathbf{M}_1$ has a rank of 1. According to Lemma 3.3, the spurious plane solutions define a consistent motion if and only if the plane normal vectors $\mathbf{L}_i^2$, i = 2, ..., n, are all parallel to each other. When this occurs, there is no way to tell if $\mathbf{M}_1$ or $\mathbf{M}_2$ contains the true solution since both of them constitute a consistent motion solution and satisfy the image data. When $\mathbf{L}_i^2$, i = 2, ..., n, are all parallel to each other, $\mathbf{M}_2$ also has a rank of 1. Therefore, if and only if both $\mathbf{M}_1$ and $\mathbf{M}_2$ have a rank of 1, a spurious solution results. Q.E.D.

For noisy data, neither of $\mathbf{M}_1$ and $\mathbf{M}_2$ may have a rank of 1. In this case, we can use the eigenvalues of $\mathbf{M}_1\mathbf{M}_1^T$ and $\mathbf{M}_2\mathbf{M}_2^T$ to determine if the solution is unique and which of $\mathbf{M}_1$ and $\mathbf{M}_2$ corresponds to the correct solution.

When the motion is uniquely determined, an optimal solution of the plane normal from $\mathbf{M}_1$ or $\mathbf{M}_2$ can be obtained. Then the motion parameters can be estimated in closed forms from either $\mathbf{K}_i$ or the original correspondence data, using the knowledge of the plane normal. However, this solution may not be the most robust since it is a stepwise linear method. A robust nonlinear long sequence algorithm that applies to general surfaces as well as planar surfaces and enforces both motion and structure consistence has been presented in [4] and [5]. Nevertheless, the theorem in this section reveals some fundamental truth about the plane motion problem and can be used to determine the uniqueness of solution.

4 Summary

In this paper we have shown an uncertain situation for the two-view plane motion problem in which an infinite number of solutions of the plane motion and structure result. We have also established necessary and sufficient conditions for determining the structure and motion of a planar surface from multiple views. The conditions obtained are algorithm-independent and apply to any number of images. These results enhance our understanding of the plane motion problem.

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